POVMs and Naimark's theorem without sums

Bob Coecke[1,2]

Oxford University Computing Laboratory,
Wolfson Building, Parks Road,
OX1 3QD Oxford, UK.

Eric Oliver Paquette[3,4]

Université de Montréal,
Laboratoire d'Informatique Théorique et Quantique,
CP 6128, succursal centre-ville, Montréal, Canada H3C 3J7.

Abstract

We introduce an abstract notion of POVM within the categorical quantum mechanical semantics in terms of †-compact categories. Our definition is justified by two facts: i. we provide a purely graphical abstract counterpart to Naimark's theorem, which establishes a bijective correspondence between POVMs and abstract projective measurements on an extended system; ii. in the category of Hilbert spaces and linear maps our definition coincides with the usual one.

Keywords: Categorical quantum mechanics, †-compact category, projective measurement, POVM, Naimark's theorem.

1 Introduction

The work presented in this paper contributes to a line of research which aims at recasting the quantum mechanical formalism in purely *category-theoretic terms* [2,3,10,19], providing it with compositionality, meaningful types, additional degrees of axiomatic freedom, a comprehensive operational foundation, and in particular, high-level mechanisms for reasoning i.e. *logic*. The computational motivation for this line of research, if not immediately obvious to the reader, can be found in earlier papers e.g. [2]. Particularly informal physicist-friendly introductions to this

[1] B.C. enjoys support from EPSRC grant EP/C500032/1 *High-level methods in quantum computation and quantum information.*

[2] Email: coecke@comlab.ox.ac.uk

[3] E.O.P. thanks Oxford University Computing Laboratory for its hospitality during his visit in which this work was realized, and for which he enjoyed financial support from Gilles Brassard's Chaire du recherche du Canada en informatique quantique. He also thanks André Méthot for fruitful feedback on an earlier version.

[4] Email: eopaquette@isodensity.ca

program are available [7,8,9]. This program originates in a paper by Samson Abramsky and one of the authors [2], and an important contribution was made by Peter Selinger, establishing an abstract definition of mixed state and completely positive map in purely multiplicative terms [19]. The starting point of this paper is a recent category-theoretic definition for projective quantum measurements which does not rely on any additive structure, due to Dusko Pavlovic and one of the authors [10]. We will refer to this manner of defining quantum measurements as *coalgebraically*. We show that the usual notion of POVM (e.g. [6,11,17]) admits a purely multiplicative category-theoretic counterpart, in the sense that it is supported both by a *Naimark-type* argument with respect to the coalgebraically defined 'projective' quantum measurements, and by the fact that we recover the usual notion of POVM when we consider the category of Hilbert spaces and linear maps.

Recall that a *projective measurement* is characterised by a set of projectors $\{\mathrm{P}_i : \mathcal{H} \to \mathcal{H}\}_i$, i.e. for all i we have $\mathrm{P}_i \circ \mathrm{P}_i = \mathrm{P}_i = \mathrm{P}_i^\dagger$, such that $\sum_i \mathrm{P}_i = 1_\mathcal{H}$, which implicitly implies that for $i \neq j$ we have $\mathrm{P}_i \circ \mathrm{P}_j = 0$. To each i we assign an *outcome probability* $\mathrm{Tr}(\mathrm{P}_i \circ \rho)$. More generally, a *POVM* is a set of positive operators $\{F_i : \mathcal{H} \to \mathcal{H}\}_i$, i.e. $F_i = f_i^\dagger \circ f_i$ for some linear operator f_i, such that $\sum_i F_i = 1_\mathcal{H}$, and to each i we now assign an *outcome probability* $\mathrm{Tr}(F_i \circ \rho)$. By positivity and by cyclicity of the trace we can rewrite this outcome probability as $\mathrm{Tr}(f_i \circ \rho \circ f_i^\dagger)$. Note that we can always choose the f_i to be the unique *positive* square root of the positive operator F_i, a fact which we will use later on. While in the case of projective measurements the state of the system undergoes a change $\rho \mapsto \mathrm{P}_i \circ \rho \circ \mathrm{P}_i$, for a POVM one typically is only concerned with the probabilities of outcomes, not the change of state i.e. no meaning is attached to $\rho \mapsto f_i \circ \rho \circ f_i^\dagger$. So the *type* of a POVM is

$$\mathsf{POVM} : \textit{quantum (mixed) n-states} \;\; \to \;\; \textit{classical (mixed) n-states}\,.$$

Using the fact that classical n-states can be represented by $[0,1]$-valued diagonal $n \times n$-matrices with trace one we can write

$$\mathsf{POVM} :: \rho \mapsto \sum_i \mathrm{Tr}(f_i \rho f_i^\dagger)|i\rangle\langle i|$$

where we used standard Dirac notation to represent the canonical projectors $\{|i\rangle\langle i|\}_i$ with respect to the computational base $\{|i\rangle\}_i$.

Example 1.1 It is possible to distinguish certain states of a system by means of a *single* POVM which could not be distinguished by a *single* projective measurement. Suppose we have a qubit which is in one of the following two *non-orthogonal* states, expressed as density matrices:

(1)
$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \rho_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

No projective measurement can *distinguish* these two states in the following sense: there exist no projective measurement of which a particular outcome reveals *with certainty* the initial state. On the other hand, the POVM

$$E_2 = \alpha\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad E_1 = 2\alpha\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_3 = 1_\mathcal{H} - E_1 - E_2$$

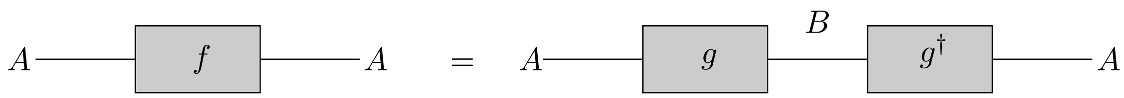
enables to distinguish these two states: observing E_1 reveals that the initial state was ρ_1 while observing E_2 reveals that it was ρ_2.

2 Abstractly defined mixed states, CPMs and projective measurements

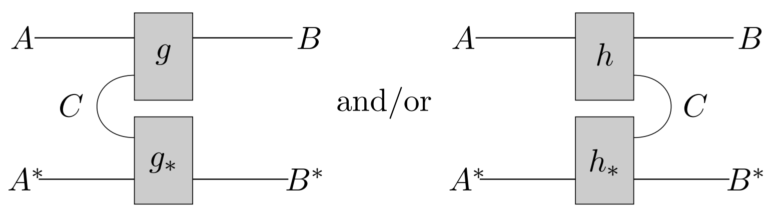
For the basic definitions of †-compact categories and their interpretation as semantics for quantum mechanics we refer to the existing literature [3,10,19] and references therein. The connection between such categories and graphical calculi is in [1,4,5,12,13,14,15,16,18,19] and references therein. We recall here the CPM-construction due to Selinger [19] and the coalgebraic characterisation of projective measurements due to Pavlovic and one of the authors [10]. This coalgebraic characterisation of projective measurements comprises the definition of *classical object* which capture the behavioural properties of classical data by making explicit the ability to copy and delete this data.

2.1 Mixed states and completely positive maps

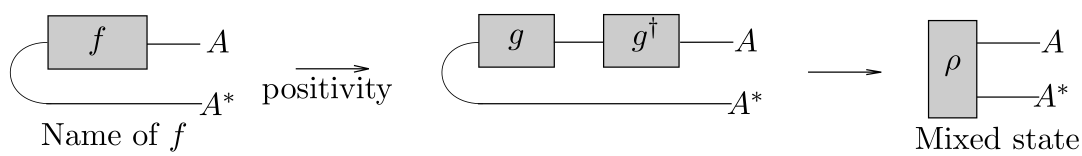
A morphism $f : A \to A$ is *positive* if there exists an object B and a morphism $g : A \to B$ such that $f = g^\dagger \circ g$. Graphically this means that we have the following decomposition:



A morphism $f : A \otimes A^* \to B \otimes B^*$ is *completely positive* if there exists an object C and morphisms $g : A \otimes C \to B$ and/or $h : A \to B \otimes C$ such that f is equal to



A *mixed state* $\rho : I \otimes I^* \to A \otimes A^*$, which is a special case of a completely positive map, is the *name* of a positive map:



— note that we rely here on the canonical isomorphism $I \simeq I \otimes I^*$.

Remark 2.1 It is worth noting that this *purely multiplicative* definition of completely positive maps (i.e. it relies on tensor-structure alone) incarnates the *Kraus representation* [17], where the usual summation is implicitly captured by the *internal trace* and/or *cotrace* on C, i.e. the half-circles in the pictures representing completely positive maps.

Given any †-compact category, define **CPM(C)** as the category with the same objects as **C**, whose morphisms $f : A \to B$ are the completely positive morphism $f : A \otimes A^* \to B \otimes B^*$ in **C**, and with composition inherited from **C**. As shown in [19], if **C** is †-compact then so is **CPM(C)**, and the morphisms of **CPM(FdHilb)** are the usual completely positive maps and mixed states. There also is a canonical 'almost' embedding of **C** into **CPM(C)** defined as

$$Pure :: f \mapsto f \otimes f_* \,.$$

From now on, we will omit $(-)^*$ on the objects and $(-)_*$ on the morphisms in the "symmetric image" which is induced by the **CPM**-construction.

2.2 *Classical objects*

The type we are after for a quantum measurement is

$$A \to X \otimes A$$

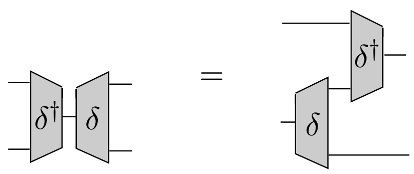
expressing that we have as input a quantum state of type A, and as output a measurement outcome of type X together with the collapsed quantum state still of type A. We distinguish between *quantum data* A and *classical data* X by our ability to freely copy and delete the latter. Hence a classical object $\langle X, \delta, \epsilon\rangle$ is defined to be an object X together with a *copying operation* $\delta : X \to X \otimes X$ and a *deleting operation* $\epsilon : X \to I$, which satisfy some obvious behavioural constraints that capture the particular nature of these operations. Let $\lambda_X : X \simeq I \otimes X$ be the natural isomorphism of the monoidal structure and let $\eta_X : I \to X^* \otimes X$ be the *unit* of the †-compact structure for object X.

Theorem 2.2 [10] *Classical objects can be equivalently defined as:*

(i) *special †-compact Frobenius algebras $\langle X, \delta, \epsilon\rangle$ which realise $\eta_X = \delta \circ \epsilon^\dagger$, where speciality means $1_X = \delta^\dagger \circ \delta$ and the †-Frobenius identity*

$$\delta \circ \delta^\dagger = (1_X \otimes \delta) \circ (\delta^\dagger \otimes 1_X)$$

depicts as



(ii) *special X-self-adjoint internal commutative comonoids $\langle X, \delta, \epsilon\rangle$, where X-self-adjointness stands for*

$$\delta = (1_X \otimes \delta^\dagger) \circ (\eta_X \otimes 1_X) \circ \lambda_X \,.$$

In particular do we have self-duality of X i.e. we can choose $X^* := X$, and also δ and ϵ prove to be self-dual i.e. $\delta_* = \delta$ and $\epsilon_* = \epsilon$.

2.3 Coalgebraically defined projective measurements

Classical objects, being internal commutative comonoids, canonically induce commutative comonads, so we can consider the Eilenberg-Moore coalgebras with respect to these. This results in the following characterization of quantum spectra as the X*-self-adjoint coalgebras* for those comonads. Given a classical object $\langle X, \delta, \epsilon\rangle$, a *projector-valued spectrum* is a morphism $\mathcal{P} : A \to X \otimes A$ which is X-complete i.e. $(\epsilon \otimes 1_A) \circ \mathcal{P} = \lambda_A$, and which also satisfies

$$\begin{tikzcd} A \arrow[r, "\mathcal{P}"] \arrow[d, "\mathcal{P}"'] & X \otimes A \arrow[d, "1_X \otimes \mathcal{P}"] \\ X \otimes A \arrow[r, "\delta \otimes 1_A"'] & X \otimes X \otimes A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\mathcal{P}"] \arrow[d, "\simeq"'] & X \otimes A \\ I \otimes A \arrow[r, "\eta_X \otimes 1_A"'] & X \otimes X \otimes A \arrow[u, "1_X \otimes \mathcal{P}^\dagger"'] \end{tikzcd}$$

to which we respectively refer as X*-idempotence* and X*-self-adjointness*.

Remark 2.3 It is most definitely worth noting that X-idempotence exactly incarnates *von Neumann's projection postulate*, in a strikingly resource-sensitive fashion: repeating a quantum measurement has the same effect as merely copying the data obtained in the first measurement.

As shown in [10], in **FdHilb** these projector-valued spectra are in bijective correspondence with the usual projector spectra defined in terms of self-adjoint linear operators. In particular, the classical object

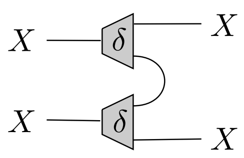
$$\left\langle \mathbb{C}^{\oplus n}\,,\ |\,i\rangle \mapsto |\,ii\rangle\,,\ |\,i\rangle \mapsto 1 \right\rangle$$

yields the projector spectra of all n-outcome measurements on a Hilbert space of dimension $k \geq n$, where X-idempotence assures projectors to be idempotent ($\mathrm{P}_i^2 = \mathrm{P}_i$) and mutually orthogonal ($\mathrm{P}_i \circ \mathrm{P}_{j\neq i} = \mathbf{0}$), X-self-adjointness assures them to be self-adjoint ($\mathrm{P}_i^\dagger = \mathrm{P}_i$), and X-completeness assures $\sum_{i=1}^{i=n} \mathrm{P}_i = 1_{\mathcal{H}}$ i.e. probabilities arising from the Born-rule add up to 1.

Given this representation theorem, and the fact that such a projector-valued spectrum already admits the correct type of a quantum measurement, one might think that projector-valued spectra are in fact quantum measurements. Unfortunately this is not the case: a projector-valued spectrum preserves the relative phases encoded in the initial state. In other words, the off-diagonal elements of the density matrix of the initial state expressed in the measurement basis do not vanish. But this can be easily fixed. In [10] it was shown that these redundant phases can be eliminated by first embedding **C** into **CPM**(**C**) and then post-composing the image $\mathcal{P}\otimes\mathcal{P}_*$ of a projector-valued spectrum $\mathcal{P}$ under *Pure* with $1_A\otimes\mathrm{Decohere}\otimes 1_A$ where

$$\mathrm{Decohere} := (1_X \otimes \eta_X^\dagger \otimes 1_X) \circ (\delta_X \otimes \delta_X) : X \otimes X \to X \otimes X$$

or, graphically,



Note that Decohere is indeed a morphism in $\mathbf{CPM}(\mathbf{C})$. One also verifies that equivalently one can set Decohere $= \delta \circ \delta^\dagger$. Conclusively, a projective measurement is a composite

$$\mathcal{M} := (1_A \otimes \text{Decohere} \otimes 1_A) \circ (\mathcal{P} \otimes \mathcal{P}_*)$$

where X carries a classical object structure and $\mathcal{P}$ is a corresponding projector-valued spectrum, and is of type $A \to X \otimes A$ in $\mathbf{CPM}(\mathbf{C})$. Note that such a fully comprehensive compositional presentation of a projective quantum measurement is not provided in the standard literature.

We will slightly relax this measurement notion by dropping X-completeness, something which is quite standard in quantum information literature where rather than $\sum_i F_i = 1_{\mathcal{H}}$ one regularly only requires $\sum_i F_i \leq 1_{\mathcal{H}}$ for POVMs. The same relaxation applies to our definition of projector-valued spectra.

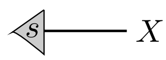
3 Abstract POVMs

In the same vein as the notions of X-self-adjointness, X-idempotence, and also X-unitarity introduced in [10], we now define the appropriate generalisations of scalars, their inverses, and positivity of morphisms.

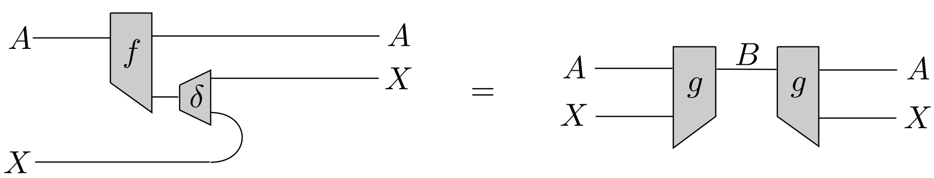
Definition 3.1 Given a classical object $\langle X, \delta, \epsilon\rangle$, an X*-scalar* is a morphism $f : I \to X$. An X-scalar $t : I \to X$ is an X*-inverse* of $s : I \to X$ iff, setting $\lambda_I : I \simeq I \otimes I$, we have

$$\delta^\dagger \circ (s \otimes t) \circ \lambda_I = \epsilon^\dagger \,.$$

In **FdHilb** X-scalars are n-tuples of complex numbers. Since $\delta^\dagger : X \otimes X \to X$ 'compares' outcomes (cf. Dirac's delta-function) an X-scalar's X-inverse in **FdHilb** is the n-tuple consisting of the component-wise inverses to the given n-tuple. In our context, X-scalars in particular arise when tracing out A in a morphism $f : A \to A \otimes X$, yielding the X-scalar $\mathrm{Tr}^A_{I,X}(f) : I \to X$. Graphically an X-scalar represents as



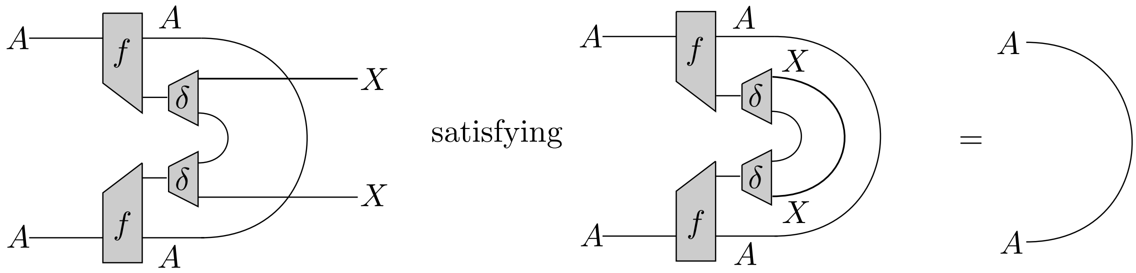
Definition 3.2 Given a classical object $\langle X, \delta, \epsilon\rangle$ a morphism $f : A \to A \otimes X$ is X*-positive* if there exists a morphism $g : B \to A \otimes X$ such that



In the second picture, the trapezoid with the corner pointing to the left indicates that the morphism it represents is equipped with the dagger as compared to the one with the corner pointing to the right — this graphical notation will be reused in what follows.

From now on, we will work within $\mathbf{CPM}(\mathbf{C})$. Classical objects will however always be defined in $\mathbf{C}$, and then embedded in $\mathbf{CPM}(\mathbf{C})$ via *Pure*.

Definition 3.3 Let $\langle X, \delta, \epsilon\rangle$ be a classical object. A *POVM* on a system of type A which produces outcomes in X is a morphism



and for which $f \in \mathbf{C}(A, X \otimes A)$ is X-positive.

Hence, within $\mathbf{CPM}(\mathbf{C})$ the type of such a POVM is indeed $A \to X$.

Theorem 3.4 *In the category* **FdHilb** *the abstract POVMs of Definition 3.3 exactly coincide with the assignments* $\rho \mapsto \sum_i \mathrm{Tr}(f_i\rho f_i^\dagger)|i\rangle\langle i|$ *corresponding to POVMs defined in the usual manner* (cf. Section 1).

Proof. Consider a POVM as in Definition 3.3. In **FdHilb** classical objects are of the form $\mathbb{C}^{\oplus n}$ and induce canonical base vectors $|\,i\rangle : \mathbb{C} \to \mathbb{C}^{\oplus n}$. Set

$$\hat{f}_i := (\langle i\,| \otimes 1_A) \circ f : A \to A \quad \text{and} \quad f_i := (|\,i\rangle\langle i\,| \otimes 1_A) \circ f : A \to X \otimes A\,.$$

In particular do we have $f = \sum_{i=1}^{i=n} f_i$. Hence we can rewrite the POVM as

$$\mathrm{tr}^A\Big[\mathrm{Decohere} \circ \Big(\sum_i f_i \otimes \sum_j f_{j*}\Big) \circ -\Big] = \mathrm{tr}^A\Big[\mathrm{Decohere} \circ \sum_{i,j}(f_i \otimes f_{j*}) \circ -\Big]$$
$$= \mathrm{tr}^A\Big[\sum_i (f_i \otimes f_{i*}) \circ -\Big]\,.$$

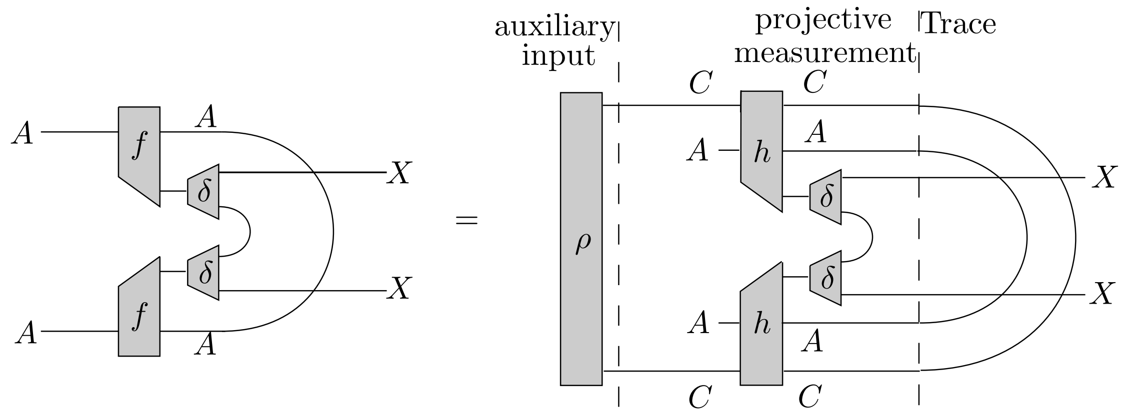
Passing from $\mathbf{CPM}(\mathbf{C})$ to standard Dirac notation, i.e. from $|\,i\rangle \otimes |\,i\rangle_*$ to $|\,i\rangle\langle i\,|$ and from $(f \otimes f_*) \circ -$ to $f(-)f^\dagger$, also using $f_i = (|\,i\rangle \otimes 1_A) \circ \hat{f}_i$, we obtain

$$\sum_i \mathrm{Tr}(\hat{f}_i(-)\hat{f}_i^\dagger)|i\rangle\langle i|$$

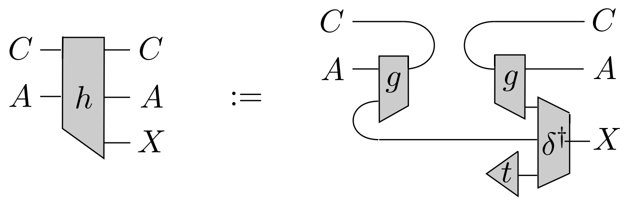
which is the intended result. Finally, the abstract normalisation condition tells us that indeed $f^\dagger \circ f = 1_A$. The converse direction constitutes analogous straightforward translation into the graphical language. □

Theorem 3.5 [Abstract Naimark theorem] *Given an abstract POVM, there exists an abstract projective measurement on an extended system which realizes this POVM. Conversely, each abstract projective measurement on an extended system yields an abstract POVM.*

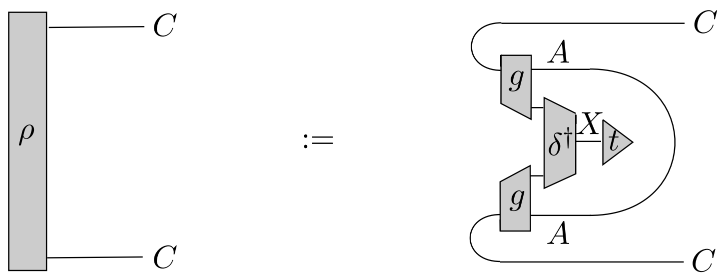
Proof: We need to show that there exists a projective measurement $h : C \otimes A \to C \otimes A \otimes X$ in $\mathbf{C}$ together with an auxiliary input $\rho : I \to C$ in $\mathbf{CPM}(\mathbf{C})$ such that they produce the same probability as a given POVM defined via $f : A \to A \otimes X$, as in Definition 3.3, provided we trace out the extended space after the measurement. Graphically this boils down to



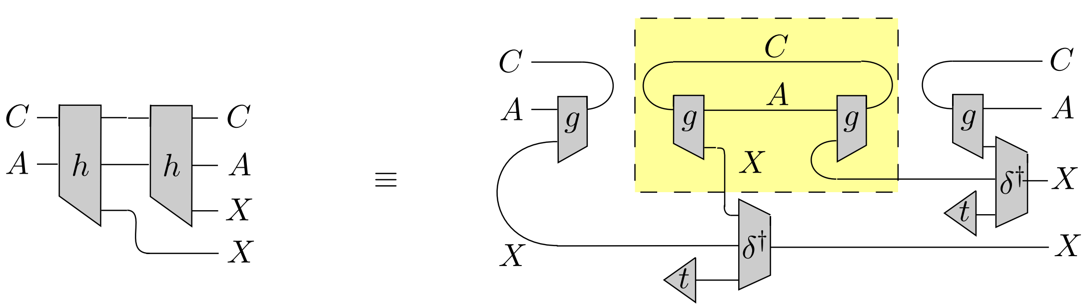
i.e. an equality between two morphisms of type $A \to X$ in $\mathbf{CPM}(\mathbf{C})$. To this end let



where $f = g^\dagger \circ g$ by X-positivity as in Definition 3.2 (we will consider g to be fixed for the reminder of the proof), where $t := \left(\mathrm{Tr}^A(f)\right)^{-1}$ is an X-scalar, [5] and where the $\delta^\dagger$ with three input wires is $(1_X \otimes \delta^\dagger) \circ \delta^\dagger$ — which is meaningful by associativity of the comultiplication. Let

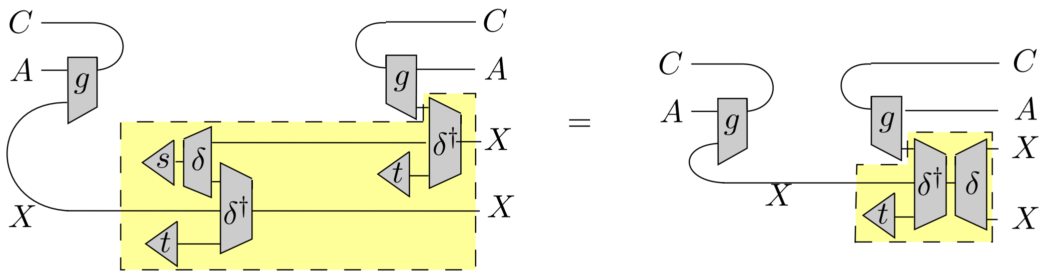


We now check X-idempotence of h. We have

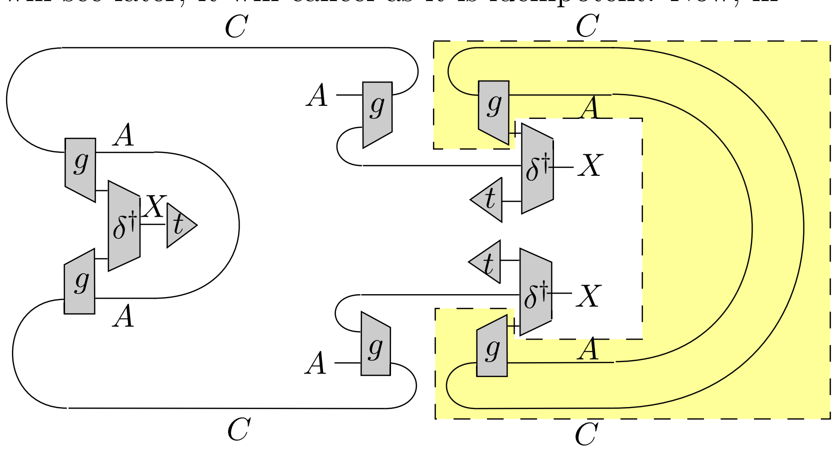


[5] It was observed by Pavlovic and one of the authors that every †-compact category $\mathbf{C}$ admits a universal *localization* $L\mathbf{C}$ together with a †-compact functor $\mathbf{C} \to L\mathbf{C}$, which is initial for all †-compact categories with †-compact functors from $\mathbf{C}$, and where a †-compact category is *local* iff all of its positive scalars are either divisors of zero, or invertible, where *zero* is multiplicatively defined in the obvious manner. These considerations extend to X-scalars. This result will appear in a forthcoming paper.

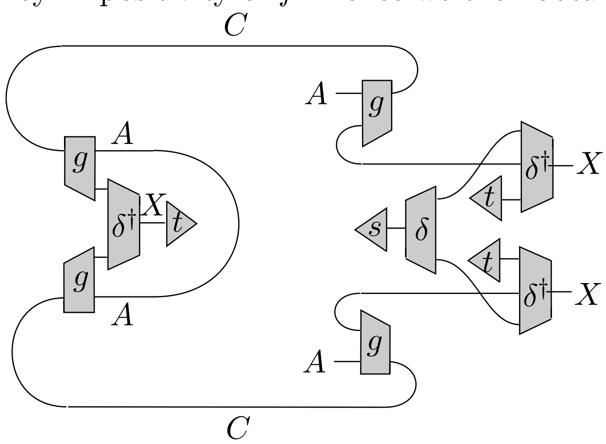
Via X-positivity of f, the pale square on the previous picture becomes $\delta \circ s$ where $s := \mathrm{Tr}^A(f)$ is an X-scalar which is inverse to the X-scalar t. Using the cancellation of relative inverse X-scalars, the fact that δ is a factor of η_X, X-self-adjointness of δ, and Frobenius law, one obtains the following equality between the pale squares below



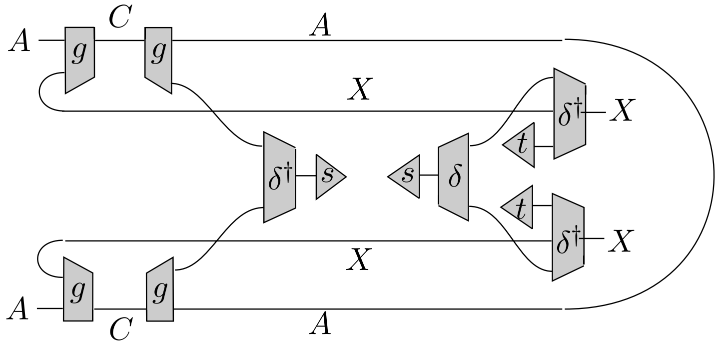
so we indeed obtain X-idempotence for h. It should be obvious that h is also X-self-adjoint by construction, so h defines a (not necessarily X-complete) projector-valued spectrum, and hence defines a projective measurement by adjoining the Decohere-morphism. Next we show that this projective measurement indeed realizes the given POVM when feeding-in the mixed state ρ, as defined above, to its C-input, and when tracing-out the A-output. In the following, we will ignore the Decohere-morphism since, as we will see later, it will cancel as it is idempotent. Now, in



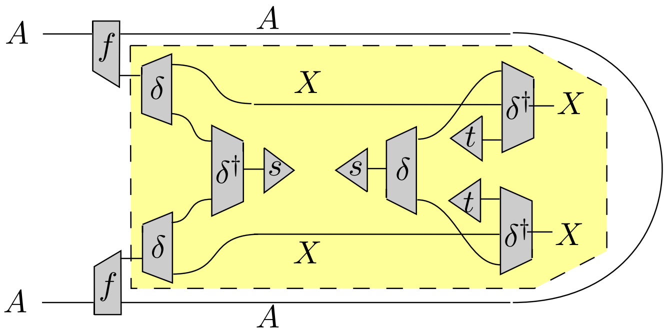
the pale square is $\delta \circ s$ by X-positivity of f. Hence we then obtain



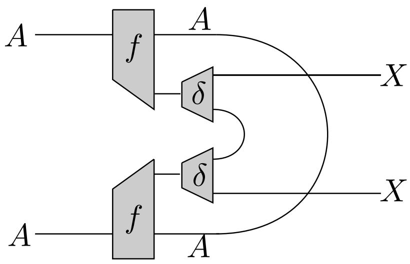
Via an obvious graph isomorphism we get



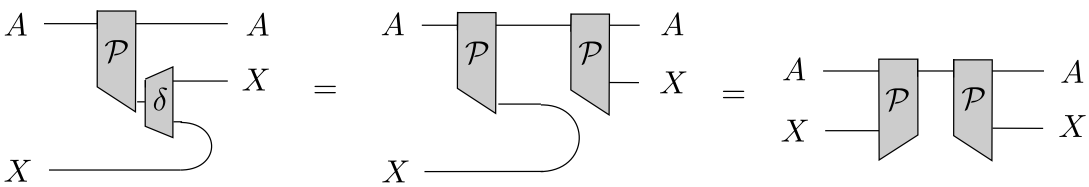
Again, by X-positivity of f, we obtain



The pale square in the previous picture reduces to the Decohere-morphism using the Frobenius law several times, cancellation of relative inverse X-scalars, etc. Readjoining the Decohere-morphism which we omitted, which now cancels out by Decohere's idempotence, we finally obtain

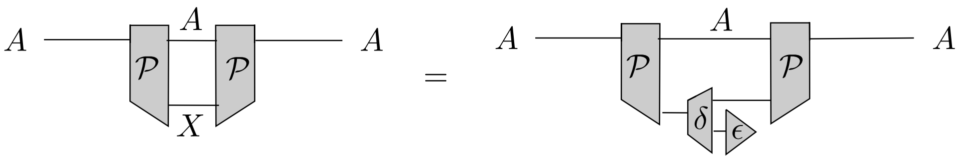


Conversely, we need to show that each projective measurement on an extended system yields a POVM. A projector-valued spectrum is X-positive since its X-idempotence and X-self-adjointness yield

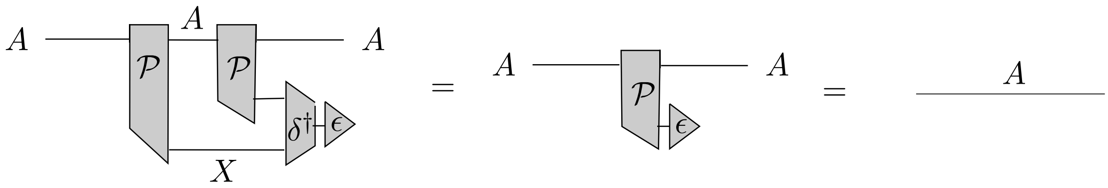


Next, observe that for an X-complete projector-valued spectrum we always have

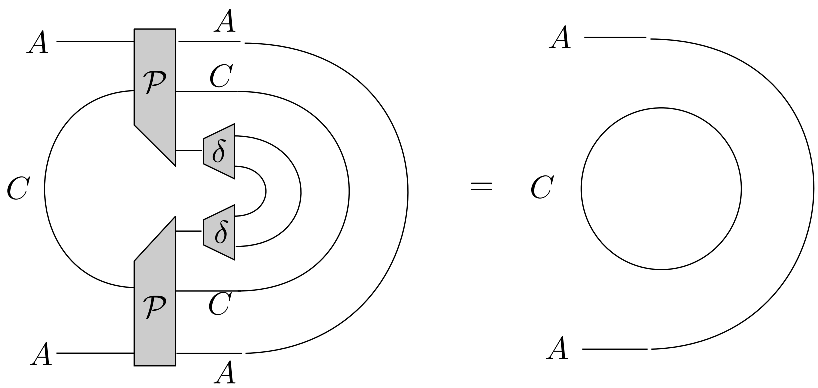
$\mathcal{P}^\dagger \circ \mathcal{P} = 1_A$ since



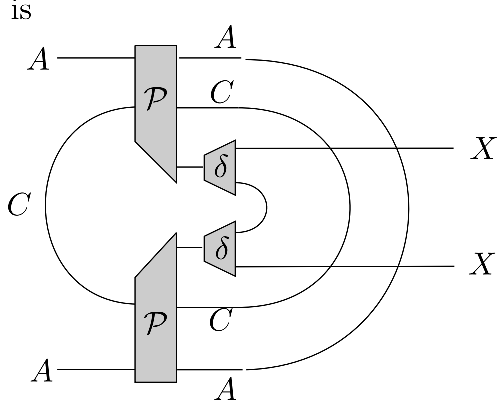
and by X-self-adjointness of $\mathcal{P}$ and δ we get



where the first equality uses X-idempotence of $\mathcal{P}$ and $\delta^\dagger \circ \delta = 1_A$. The second equality is obtained from the definition of X-completeness. Now, when considering a projective measurement on an extended system, using this fact together with $\delta^\dagger \circ \delta = 1_X$ we obtain



thence satisfying the normalization condition up to a C-dependent scalar. The POVM which we obtain is



what completes the proof. $\square$

Remark 3.6 The more complicated manipulations in the above proof concern classical data, while the quantum data manipulations are quite canonical. The fact that the classical data manipulations are sophisticated is due to the explicit resource-sensitive account on classical data.

Remark 3.7 While POVMs are not concerned with the state after the measurement, our analysis does produce an obvious candidate for non-destructive generalised measurements, sometimes referred to as PMVMs in the literature [11]. We postpone a discussion to forthcoming writings.

Remark 3.8 Notice the delicate role which X-completeness and normalisation of the POVMs plays in all this, on which, due to lack of space, we cannot get into. We postpone this discussion to an extended version of the present paper, which is forthcoming.

References

[1] S. Abramsky (2005) *Abstract scalars, loops, free traced and strongly compact closed categories*. In: Proceedings of CALCO 2005, pp. 1–31, Springer Lecture Notes in Computer Science **3629**.

[2] S. Abramsky and B. Coecke (2004) *A categorical semantics of quantum protocols*. In: Proceedings of the 19th IEEE Conference on Logic in Computer Science, pp. 415–425, IEEE Computer Science Press. E-print arXiv:quant-ph/0402130.

[3] S. Abramsky and B. Coecke (2005) *Abstract physical traces*. Theory and Applications of Categories **14**, pp. 111–124. Available from `www.tac.mta.ca/tac/volumes/14/6/14-06abs.html`.

[4] J. Baez (2004) *Quantum quandaries: a category-theoretic perspective*. In: Structural Foundations of Quantum Gravity, Oxford University Press. E-print arXiv:quant-ph/0404040.

[5] J. Baez and J. Dolan (1995) *Higher-dimensional algebra and topological quantum field theory*. Journal of Mathematical Physics **36**, pp. 6073–6105. E-print arXiv:q-alg/9503002.

[6] P. Busch, P. J. Lahti and P. Mittelstaedt (1991) *The Quantum Theory of Measurement*. Springer Lecture Notes in Physics **2**.

[7] B. Coecke (2005) *Kindergarten quantum mechanics — lecture notes*. In: Quantum Theory: Reconsiderations of the Foundations III, pp. 81–98, AIP Press. E-print arXiv:quant-ph/0510032.

[8] B. Coecke (2005) *Quantum information-flow, concretely, and axiomatically*. In: Proceedings of Quantum Informatics 2004, pp. 15–29, Proceedings of SPIE Vol. 5833. E-print available from arXiv:quant-ph/0506132.

[9] B. Coecke (2006) *Introducing categories to the practicing physicist*. In: What is Category Theory? Advanced Studies in Mathematics and Logic **30**, pp. 45–74, Polimetrica Publishing.

[10] B. Coecke and D. Pavlovic (2006) *Quantum measurements without sums*. To appear in: Mathematics of Quantum Computing and Technology. Taylor and Francis. Forthcoming on arXiv:quant-ph.

[11] E. B. Davies (1976) *Quantum Theory of Open Systems*. Academic Press.

[12] P. Freyd and D. Yetter (1989) *Braided compact closed categories with applications to low-dimensional topology*. Advances in Mathematics **77**, pp. 156–182.

[13] A. Joyal and R. Street (1991) *The geometry of tensor calculus I*. Advances in Mathematics **88**, pp. 55–112.

[14] A. Joyal, R. Street and D. Verity (1996) *Traced monoidal categories*. Proceedings of the Cambridge Philosophical Society **119**, pp. 447–468.

[15] G. M. Kelly (1972) *Many-variable functorial calculus I*. In: Coherence in Categories, pp.66–105, Springer Lecture Notes in Mathematics **281**.

[16] G. M. Kelly and M. L. Laplaza (1980) *Coherence for compact closed categories*. Journal of Pure and Applied Algebra, **19**, pp. 193–213.

[17] K. Kraus (1983) *States, Effects, and Operations*. Springer-Verlag.

[18] R. Penrose (1971) *Applications of negative dimensional tensors*. In: Combinatorial Mathematics and its Applications, pp. 221–244, Academic Press.

[19] P. Selinger (2006) *Dagger compact closed categories and completely positive maps*. To appear in: Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005), Electronic Notes in Theoretical Computer Science.