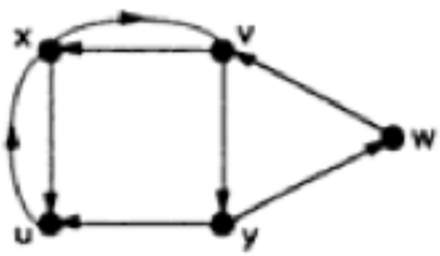
MATH 3330: Applied Graph Theory

ASSIGNMENT #2 *SOLUTIONS*

1. For the graph shown below and the given vertex sequences (i-iv),



   i) <x,v,y,w,v>
   ii) <x,u,x,u,x>
   iii) <x,u,v,y,x>
   iv) <x,v,y,w,v,u,x>

   a) Which of the vertex sequences represent a directed walk in the graph?

   walk: an alternating sequence of vertices and edges, representing a continuous traversal from the v_0 to v_n.

   The directed walks are (i) and (ii). Not (iii), because there is no arc from u to v. Not (iv), because there is no arc from v to u.

   b) What are the lengths of those that are directed walks?

   The lengths of both directed walks, (i) and (ii), is 4.

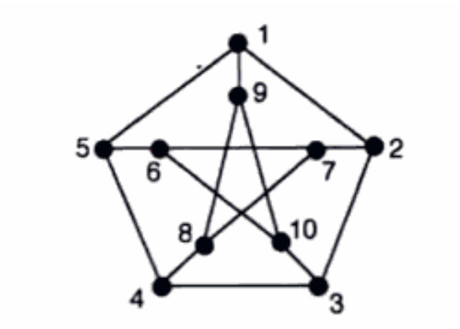
   c) Which directed walks are directed paths?

   Neither (i) nor (ii) are directed paths because both repeat vertices. Walk (i) repeats vertex v and walk (ii) repeats vertex u.

   d) Which directed walks are directed cycles?

Neither are directed cycles: (i) does not start and end at the same vertex and, while <x,u,x> is a directed cycle, (ii) repeats vertices/edges and so is not a directed cycle.

2. In the Petersen graph shown below,



a) Find a trail of length 5.

trail: an alternating sequence of vertices and edges with no repeated edges.

In the Petersen graph, there are several trails of length 5.
For example: <1,2,3,4,5,6>

b) Find a path of length 9.

path: a trail with no repeated vertices (except possibly the initial and final vertex).

Again, there are several. For example: <1,2,3,4,5,6,7,8,9,10>

c) Find cycles of length 5, 6, 8 and 9.

cycle: a closed path.

For each cycle length, there are many possibilities. Here are a few.
C_5 = *<1,2,3,4,5,1> or <6,7,8,9,10,6> or <1,2,7,8,9,1>*
C_6 = *<1,2,3,4,8,9,1>*
C_9 = *<1,2,3,4,5,6,7,8,9,1>*

3. Determine the girth of the graphs indicated:

a) Complete bipartite graph $K_{m,n}$ for $m \geq n \geq 3$,

girth: length of the shortest cycle in the graph.

In the complete bipartite graph, there exist no cycles of length 1 or 2 (since no self-loops or multi-edges), and no cycles of length 3 (since bipartite. However, even for $m \geq n \geq 2$, there exists a cycle of length 4. Thus, $K_{m,n}$ for $m \geq n \geq 3$ have girth 4.

b) Complete graph K_n $n \geq 3$, and

Again, there are no self-loops nor multi-edges, so no cycles of length 1 or 2. Since all vertices in a K_n are mutually adjacent, and $n \geq 3$, there does exists a cycle of length 3. Thus, K_n for $n \geq 3$ have girth 3.

c) The Petersen graph.

One can check that there are no cycles of length 3 or 4 in the Petersen graph. However, we know from (2) that there are several cycles of length 5. Thus, the Petersen graph has girth 5.

4. Determine whether the Petersen graph is hamiltonian.

hamiltonian cycle: cycle that uses every vertex of a graph.
hamiltonian graph: graph that has a hamiltonian cycle.

There exists a hamiltonian path (see 2b), but no hamiltonian cycle. Thus, the Petersen graph is not hamiltonian.

However, it is interesting to note that by deleting any vertex in the Petersen graph, it makes it hamiltonian.

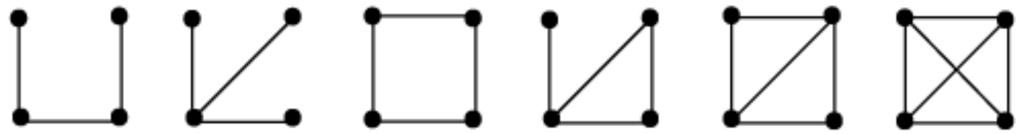
5. Give the number of different eulerian tours in K_4.

eulerian trail: a trail that contains every edge of the graph.
eulerian tour: a closed eulerian trail.

There are zero. There is no closed

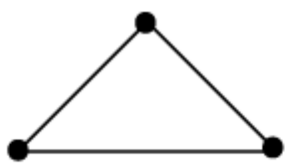
6. Find all possible isomorphism types of a simple connected graph with 4 vertices.

   simple graph: graph with no multi-edges or self-loops.
   connected: there exists a walk between every pair of distinct vertices.



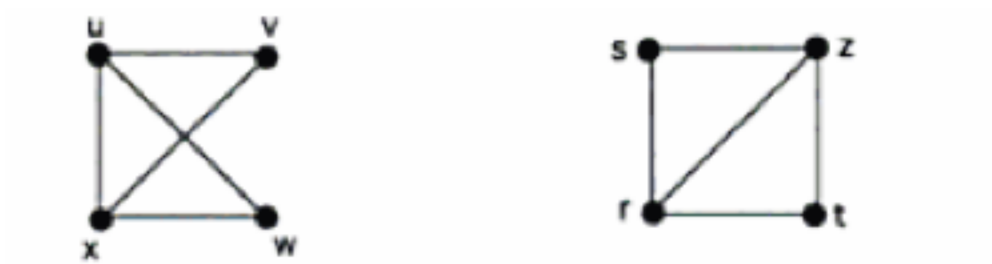
7. Find all possible isomorphism types of a simple graph with 3 vertices and 3 edges.

   Simple graph, so no self-loops or multi-edges.



8. For the following, find a vertex-bijection that specifies an isomorphism between the two graphs shown.

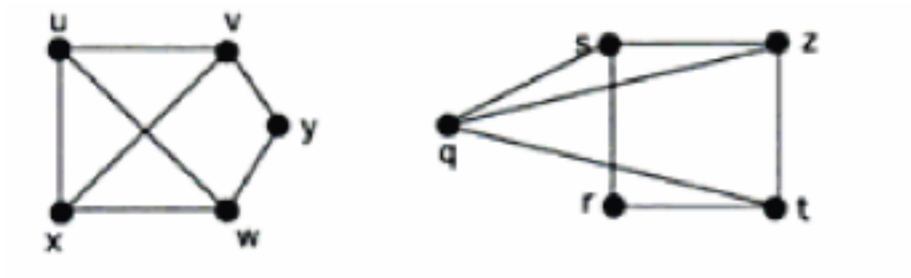
   a)



There are four possible vertex-bijections (f_1, f_2, f_3 and f_4 below) that specify an isomorphism.

$f_1(u)=r$	$f_2(u)=r$	$f_3(u)=z$	$f_4(u)=z$
$f_1(v)=s$	$f_2(b)=t$	$f_3(b)=s$	$f_4(v)=t$
$f_1(w)=t$	$f_2(w)=s$	$f_3(w)=t$	$f_4(w)=s$
$f_1(x)=z$	$f_2(x)=z$	$f_3(x)=r$	$f_4(x)=t$

   b)

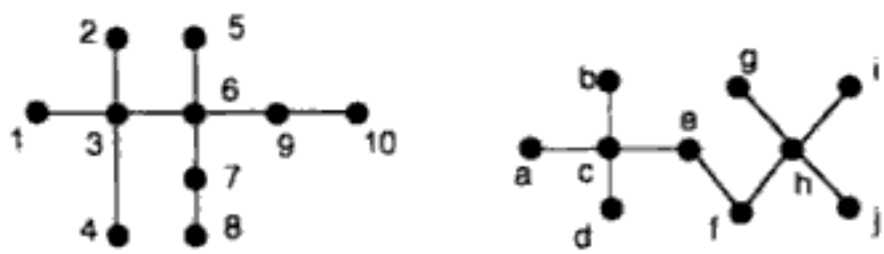


Again, there are four possible vertex-bijections (f_1, f_2, f_3 and f_4 below) that specify an isomorphism.

$f_1(u)=q$	$f_2(u)=q$	$f_3(u)=z$	$f_4(u)=z$
$f_1(v)=s$	$f_2(b)=t$	$f_3(b)=s$	$f_4(v)=t$
$f_1(w)=t$	$f_2(w)=s$	$f_3(w)=t$	$f_4(w)=s$
$f_1(x)=z$	$f_2(x)=z$	$f_3(x)=q$	$f_4(x)=q$
$f_1(y)=r$	$f_2(y)=r$	$f_3(y)=r$	$f_4(y)=r$

9. For each of the following, determine whether the graphs or digraphs in the given pair are isomorphic.

a)



These graphs are not isomorphic.

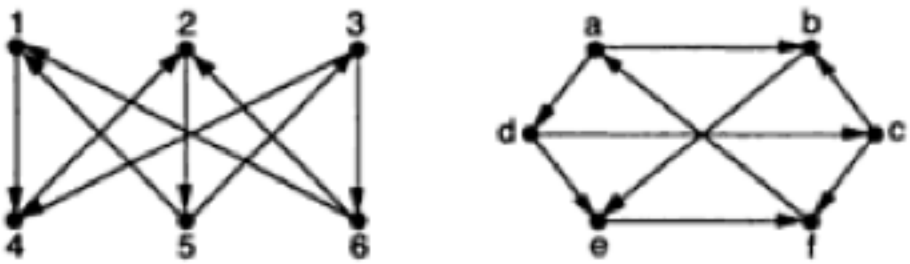
Vertex 3 (degree 4) is forced to map to either vertex c or h as they are the only two vertices in that graph with degree 4. However, the neighbors of vertex 3 have degrees 1,1,1 and 4, while the neighbors of both vertex c and h have degrees 1,1,1 and 2.

b)

These graphs are not isomorphic.

The first graph has degree sequence <3,3,3,3,2,2> and the second has degree sequence <4,3,3,2,2,2>.

c)



These graphs are isomorphic.

One vertex-bijection that specifies this isomorphism is given below:

f(1)=b
f(2)=f
f(3)=d
f(4)=e
f(5)=d
f(6)=c