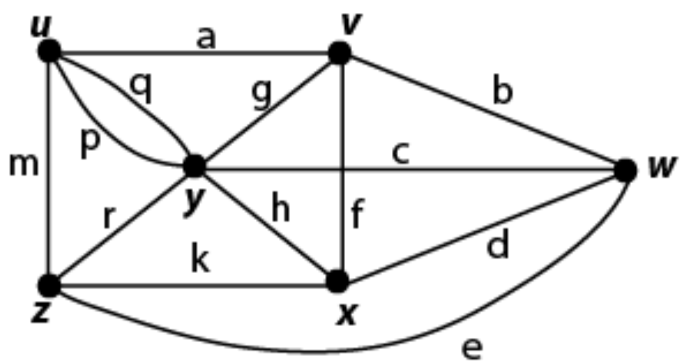
MATH 3330: Applied Graph Theory

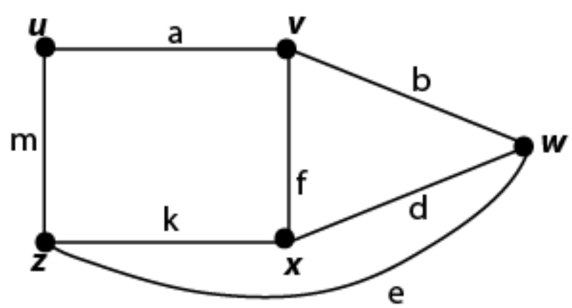
ASSIGNMENT #4 *SOLUTIONS*

1. *In the following graph,*

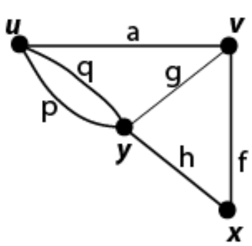


a) *Find the indicated vertex-deletion subgraphs:*

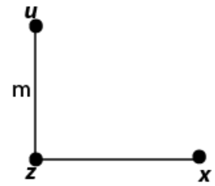
i) $G - y$



ii) $G - \{w,z\}$

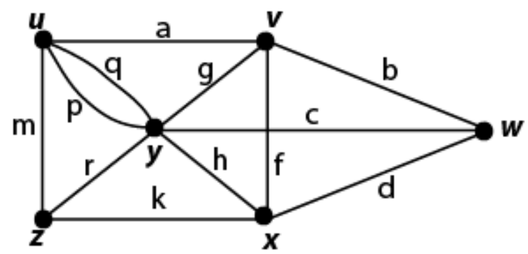


iii) $G - \{w,v,y\}$

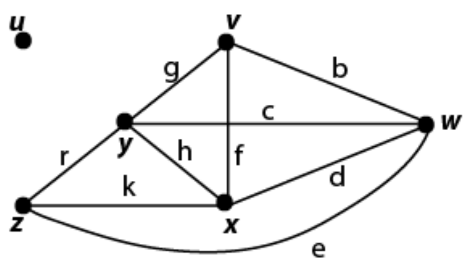


b) *Find the indicated edge-deletion subgraphs:*

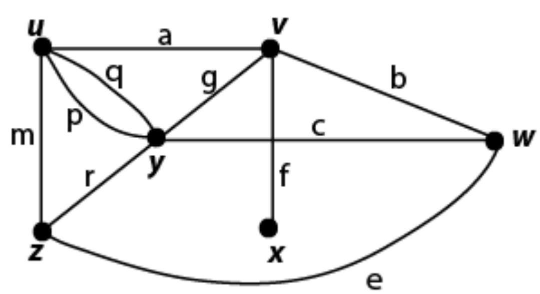
i) $G - e$



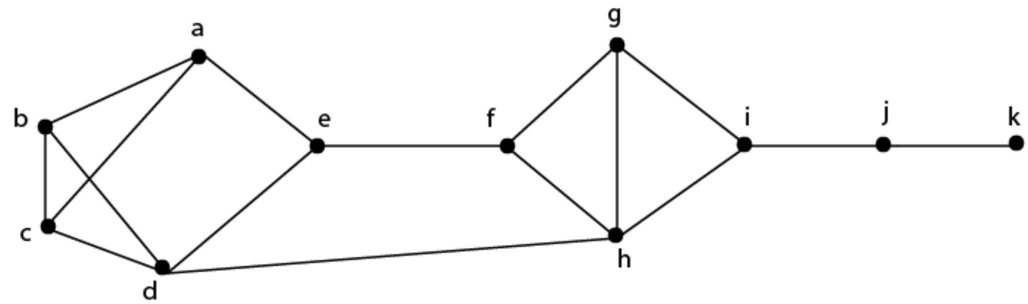
ii) $G - \{a,m,p,q\}$



iii) $G - \{d,h,k\}$



2. *In the following graph,*



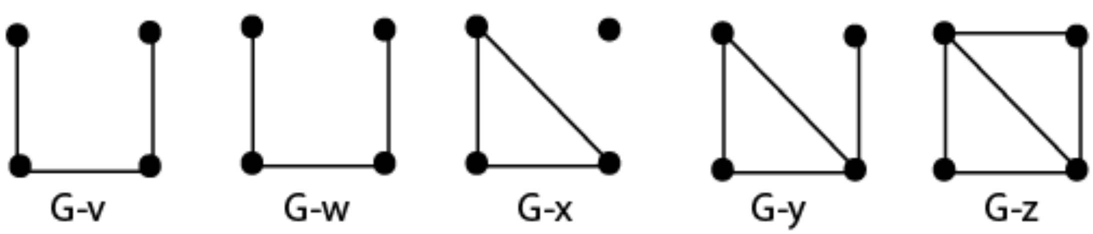
a) *Find all the cut-vertices.*

Cut-vertices are i and j.

b) *Find all the cut-edges (Give edges by its endpoints, i.e. {a,b} for the edge with endpoints a and b).*

Cut-edges are {i,j} and {j,k}. (All other edges are cycle-edges.)

3. *Find a graph with the vertex-deletion subgraph list given below.*



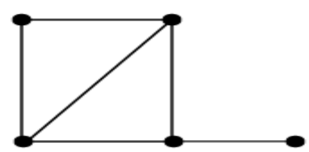
First note that since the list of the vertex-deletion subgraph list is 5, then $|V_G|=5$. By theorem 2.4.4 (given in presentation on the Graph Reconstruction Problem in class), we know that

$$\left|E_G\right| = \frac{1}{5-2}\sum_{v\in V}\left|E_{G-v}\right| = \frac{1}{3}\left(\left|E_{G-v}\right| + \left|E_{G-w}\right| + \left|E_{G-x}\right| + \left|E_{G-y}\right| + \left|E_{G-z}\right|\right) = \frac{1}{3}(3+3+3+4+5) = \frac{1}{3}(18) = 6$$

And, for each card, the degree of the missing vertex is the difference between $|E_G|$ and the number of edges on that card. Thus,

$\deg(v) = |E_G| - |E_{G-v}| = 6-3=3.$
$\deg(w) = |E_G| - |E_{G-w}| = 6-3=3.$
$\deg(x) = |E_G| - |E_{G-x}| = 6-3=3.$
$\deg(y) = |E_G| - |E_{G-y}| = 6-4=2.$
$\deg(z) = |E_G| - |E_{G-z}| = 6-5=1.$

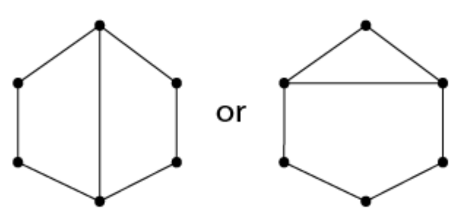
Thus, the the graph has degree sequence $\langle 3,3,3,2,1 \rangle$ and is isomorphic to



4. *True or false: The endpoints of a cut-edge are both cut-vertices. If false, explain why.*

   False. Counterexample: Take any tree of at least two vertices. Every edge is a cut-edge. However, the leaves are not cut-vertices. So any edge with a leaf as an endpoint is a counterexample.

5. *Draw a 6-vertex connected graph that has exactly seven edges and exactly three cycles.*



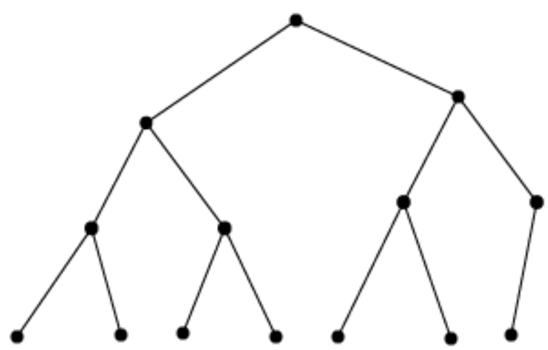
6. *True or false: There exists a connected n-vertex simple graph with n +1 edges that contains exactly 2 cycles. If true, give an example. If false, explain why not.*

   For $1 \leq n \leq 3$, this cannot be true. To even contain n+1 edges, the graphs would not be simple.

   However, for $n \geq 4$, it is TRUE! One realization of this follows: Consider the path on n vertices, P_n. We know that it has n-1 edges. Now add edges between each leaf and another non-adjacent vertex that is not the other leaf (in the example below, the vertex two away). There are exactly 2 cycles and we've added 2 edges to a P_n, so we now have (n-1)+2=n+1 edges.

7. *For the following, draw the specified tree or explain why no such tree can exist.*

   a) *A 14-vertex binary tree of height 3.*



   b) *A 16-vertex binary tree of height 3.*

   No such tree can exist. We know (by Thm. 3.2.3) that for a binary tree $h+1 \leq |V| \leq \frac{2^{h+1}-1}{2-1}$ where h is the height of the binary tree. In our case, we are looking for a tree of height 3, so $4 \leq |V| \leq \frac{2^4-1}{2-1} = \frac{16-1}{1} = 15$. So, for a binary tree to be of height 3, it may contain at most 15 vertices. Thus, a 16-vertex binary tree of height 3 cannot exist.

c) *A ternary tree of height 3 with exactly four vertices.*

There is only one, and it looks like this:

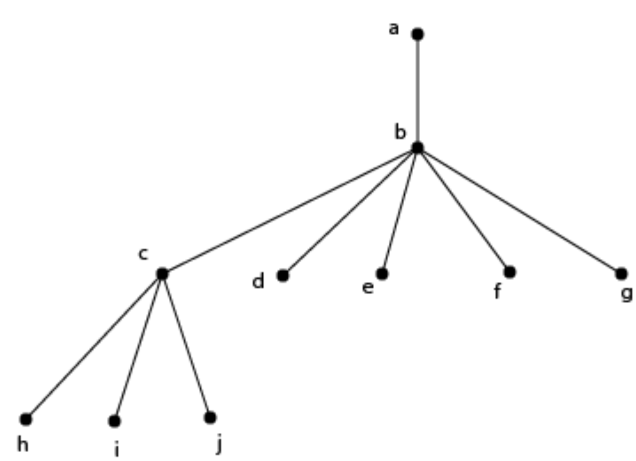
8. *What is the relationship between the depth of a vertex v in a rooted tree and the number of ancestors of v? Explain your answer.*

They are equal. The depth of a vertex is the distance to the root. An ancestor of a vertex *v* is a vertex on the path from *v* to the root. Since there is only one unique path between any two vertices, there is only one path from *v* to the root so this must be the shortest path. So the number of vertices (not *v*) on this path is the number of ancestors, and the length of this path is the depth. They must be equal.

9. *For the following, draw the root tree specified by the given array of parents.*

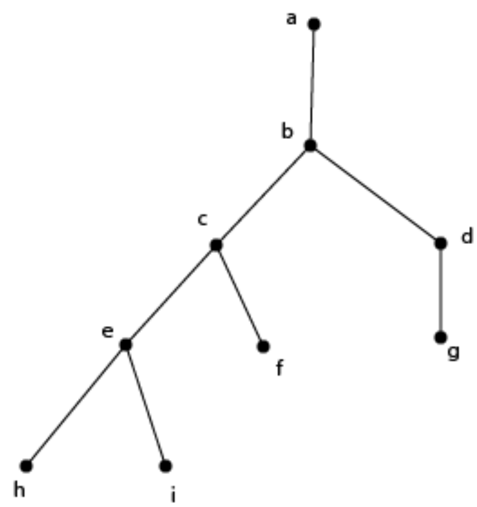
a)

Fvertex	a	b	c	d	e	f	g	h	i	j
parent	-	a	b	b	b	b	b	c	c	c

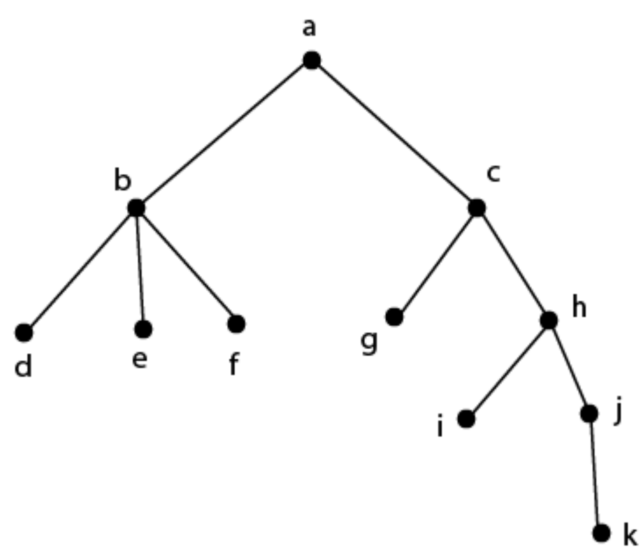


b)

vertex	a	b	c	d	e	f	g	h	i
parent	-	a	b	b	c	c	d	e	e



10. Specify the rooted tree drawn below with an array of parents.



Array of parents given by:

vertex	a	b	c	d	e	f	g	h	i	j	k
parent	-	a	a	b	b	b	c	c	h	h	j