

Chapter 4

Power Spectral Density

Let $\{X_n\}$ denote a weakly stationary random process with an autocovariance function that dies out “sufficiently quickly” in the sense¹

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty \quad (4.1)$$

If (4.1) is satisfied,

$$h(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-i\omega k} \quad -\pi \leq \omega \leq \pi \quad (4.2)$$

exists and is called the “power spectral density”.

It is straightforward to show (make sure you can)²

$$\gamma(k) = \int_{-\pi}^{\pi} h(\omega) e^{i\omega k} d\omega \quad k = 0, \pm 1, \pm 2 \dots \quad (4.3)$$

$\gamma(k)$ and $h(\omega)$ are a “Fourier Transform pair”. Given one we can recover the other. They provide complementary, but equivalent, information on $\{X_n\}$ in the lag (k) and frequency (ω) domains.

¹ $\sum_1^{\infty} k^{-2}$ is convergent and $\sum_1^{\infty} k^{-1}$ is divergent. Thus k^{-1} does not die out quickly enough with increasing k for the sum to exist. An example of such a stationary random process is the harmonic process.

²Multiply (4.2) by $e^{i\omega l}$, and integrate from $-\pi$ to π , to give

$$\int_{-\pi}^{\pi} h(\omega) e^{i\omega l} d\omega = \sum_{k=-\infty}^{\infty} \gamma(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\omega(k-l)} d\omega = \sum_{k=-\infty}^{\infty} \gamma(k) \delta_{kl} = \gamma(l)$$

Setting $k = 0$ in (4.3) gives the following expression for the variance of a process in terms of its power spectral density:

$$\sigma_X^2 = \int_{-\pi}^{\pi} h(\omega) d\omega \quad (4.4)$$

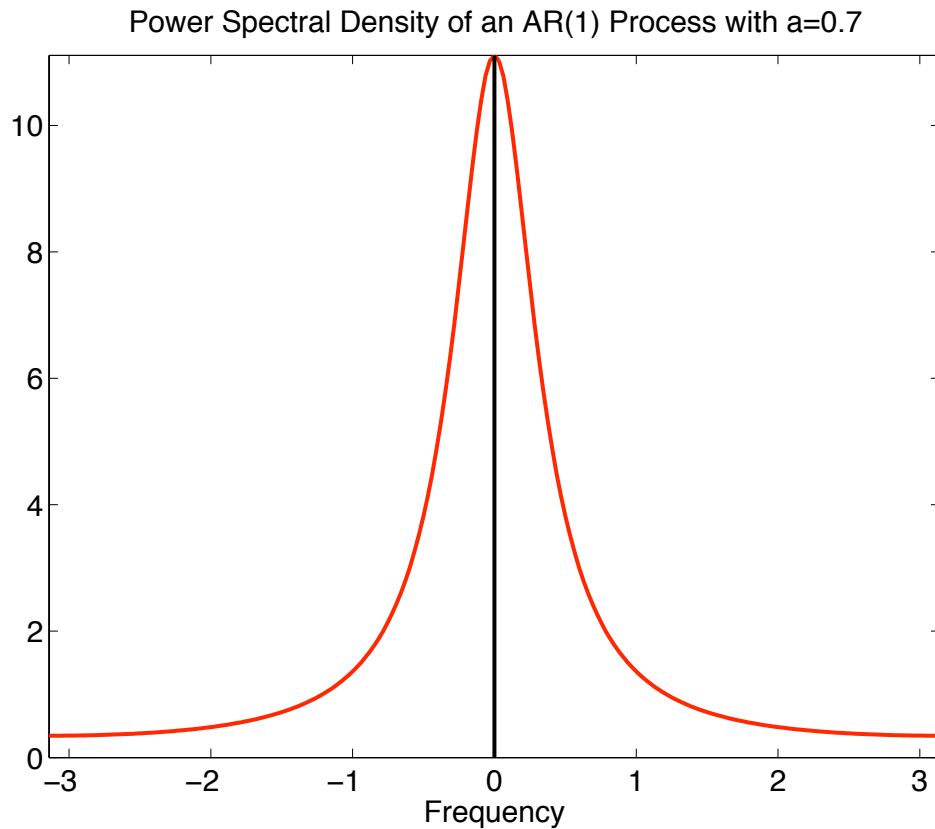


Figure 4.1: Power spectral density of an AR(1) process with $a = 0.7$. Frequency ω ranges from $-\pi$ to π . (Note that a frequency of π corresponds to a period of 2Δ .)

We will see that if $0 \leq \omega_1 < \omega_2 \leq \pi$ then $2 \int_{\omega_1}^{\omega_2} h(\omega) d\omega$ is the contribution to the variance from variations with frequencies between ω_1 and ω_2 .

SPECTRAL DENSITY GIVES A BREAKDOWN OF VARIANCE WITH RESPECT TO FREQUENCY.

4.1 Interpreting Power Spectral Density

One way to interpret power spectral density is based on the “finite Fourier transform” of X_{-N}, \dots, X_N :

$$\zeta(\omega_k) = \frac{\sqrt{\Delta\omega}}{2\pi} \sum_{n=-N}^N e^{-i\omega_k n} X_n \quad (4.5)$$

$$\Delta\omega = \frac{2\pi}{2N+1}, \quad \omega_k = k \Delta\omega = \frac{2\pi}{(2N+1)/k} \quad (4.6)$$

It is straightforward to recover the X from the ζ :³

$$X_n = \sum_{k=-N}^N e^{i\omega_k n} \zeta(\omega_k) \sqrt{\Delta\omega} \quad (4.7)$$

Thus the finite Fourier transform, and its inverse, allows us to move between the time and frequency domains with no loss of information:

$$\{X_{-N} \dots X_0 \dots X_N\} \longleftrightarrow \{\zeta_{-N} \dots \zeta_0 \dots \zeta_N\}$$

Any complex number can be written in polar form⁴ and so

$$\zeta(\omega_k) = |\zeta(\omega_k)| e^{i\phi_k}$$

and thus (4.7) can be written as the following sum of cosines:

$$X_n = \sum_{k=-N}^N \cos(\omega_k n + \phi_k) |\zeta(\omega_k)| \sqrt{\Delta\omega} \quad (4.8)$$

Question: How many amplitudes and phases are needed to recover the $2N+1$ X values?

³Multiply (4.5) by $\exp(i\omega_k m)$ and sum from $k = -N$ to N (make sure you can do this)

⁴ $z = a + ib = |z|e^{i\theta}$

If $E(X_n) = 0$, the standard estimator for the variance is

$$s_X^2 = \frac{1}{2N+1} \sum_{n=-N}^N X_n^2 \quad (4.9)$$

If you use (4.8) in (4.9) it is easy to show

$$s_X^2 = \sum_{k=-N}^N |\zeta(\omega_k)|^2 \Delta\omega \quad (4.10)$$

This is Parseval's relation and shows the sample variance can be broken down into contributions from variations at different frequencies.

To relate variance and power spectral density, take expectations of both sides of (4.10) to give

$$\sigma_X^2 = \sum_{k=-N}^N E(|\zeta(\omega_k)|^2) \Delta\omega \quad (4.11)$$

It can be shown⁵

$$E(|\zeta(\omega_k)|^2) = \int_{-\pi}^{\pi} F_N(\theta - \omega_k) h(\theta) d\theta \quad (4.12)$$

where F_N is the “Fejer kernel” (Priestley, p400) defined by

$$F_N(\theta) = \frac{1}{2\pi} \sum_{s=-N}^N \left[1 - \frac{|s|}{N} \right] \cos(s\theta)$$

The Fejer kernel is like a “spectral window” through which we view the power spectral density.

⁵To relate $E(|\zeta(\omega_k)|^2)$ to $h(\omega)$ take expectations of

$$|\zeta(\omega_k)|^2 = \zeta(\omega_k) \zeta(\omega_k)^* = \frac{\Delta\omega}{4\pi^2} \sum_{n=-N}^N e^{-i\omega_k n} X_n \sum_{m=-N}^N e^{i\omega_k m} X_m$$

The Fejer kernel is plotted in Figure 4.2. Note that as $N \rightarrow \infty$ the Fejer kernel tends to a delta function and thus $E(|\zeta(\omega_k)|^2)$ tends to $h(\omega)$. From (4.8) we see that the power spectral density tells us about the expected magnitude of the sinusoids making up the time series. From (4.11) we see that power spectral density provides a breakdown of the variance with respect to frequency.

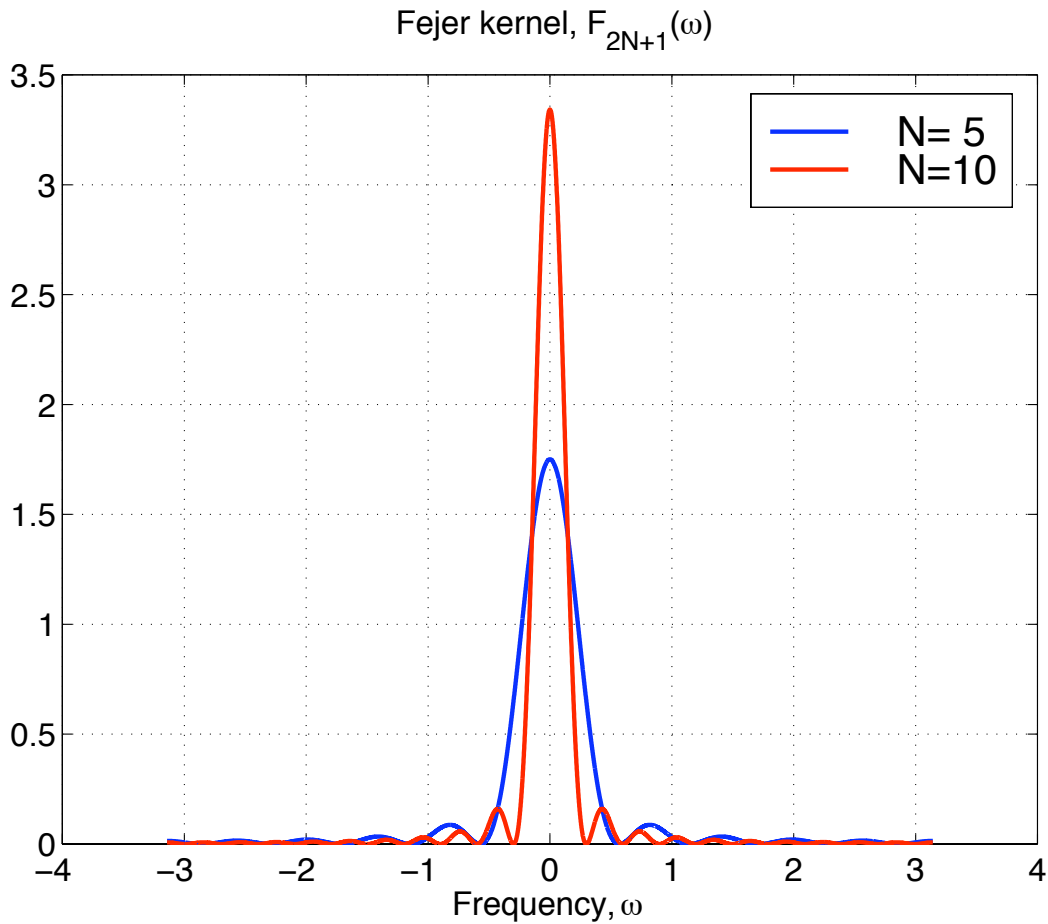


Figure 4.2: Fejer kernel for $N = 5$ and 10 . The first node is at $\omega = 2\pi/N$ and gives a measure of the width of this “spectral window”. As $N \rightarrow \infty$ the window becomes narrower and the Fejer kernel approaches a delta function.

Also note that as $N \rightarrow \infty$ the ω_k become dense over $(-\pi, \pi)$ and the sum (4.11) approaches the integral (4.4).

Let's find the power spectral densities of a couple of simple stationary random processes.

PURELY RANDOM PROCESS: Show that the power spectral density is

$$h(\omega) = \frac{\sigma^2}{2\pi}$$

THE AR(1) PROCESS: The process is defined by

$$X_n = aX_{n-1} + \epsilon_n$$

where $\{\epsilon_n\}$ is a purely random process with variance σ^2 . Show

$$h(\omega) = \frac{\sigma^2}{2\pi} |1 - ae^{-i\omega}|^{-2}$$

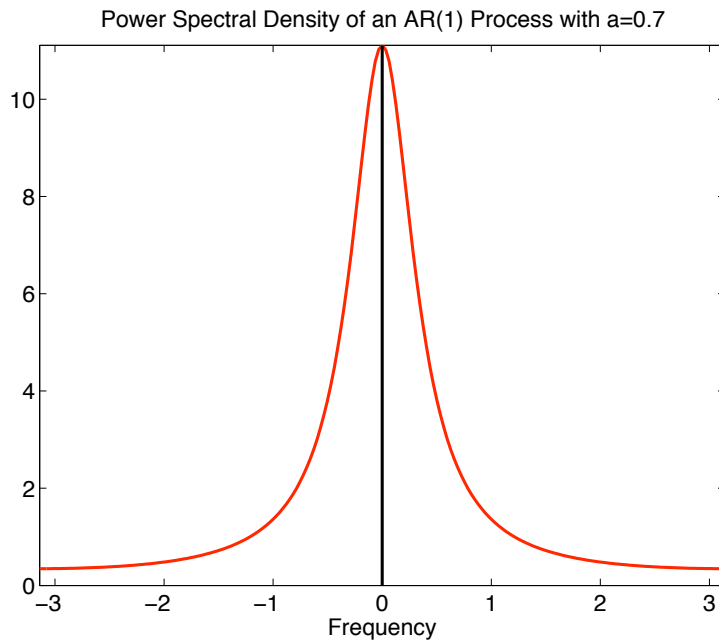


Figure 4.3: Power spectral density of an AR(1) process with $a = 0.7$. Frequency ω ranges from $-\pi$ to π . How would this plot change if $a = -0.7$ and what would typical realizations look like?

4.2 Spectral Representation

The spectral representation is a very useful device for describing stationary random processes in the frequency domain and calculating quantities like power spectra and cross spectra. It's worth learning what it means and how to use it.

We have seen that $\{X_{-N} \dots X_N\}$ can be written in the form

$$X_n = \sum_{k=-N}^N e^{i\omega_k n} \zeta(\omega_k) \sqrt{\Delta\omega} \quad n = -N, \dots, N \quad (4.13)$$

Consider the limit of (4.13) as $N \rightarrow \infty$. The ω_k will become denser over the interval $(-\pi, \pi)$ and we can envision (4.13) being replaced by an integral of the form

$$X_n = \int_{-\pi}^{\pi} e^{i\omega n} dZ(\omega)$$

where $dZ(\omega)$ plays the role of $\zeta(\omega_k) \sqrt{\Delta\omega}$. This idea leads to an extremely important and useful representation of a stationary random process. It is summarized in the following theorem (see page 536, Appendix C of Shumway and Stoffer) which describes the **SPECTRAL REPRESENTATION OF STATIONARY RANDOM PROCESSES**:

Let $\{X_n\}$ denote a zero-mean stationary random process. Then there exists a random orthogonal increment process $dZ(\omega)$ defined on $(-\pi, \pi)$ such that

$$X_n = \int_{-\pi}^{\pi} e^{i\omega n} dZ(\omega)$$

The random orthogonal increment process $dZ(\omega)$ has zero mean, $E(|dZ(\omega)|^2) = h(\omega)d\omega$ and $E[dZ(\omega_1)dZ^*(\omega_2)] = 0$ for $\omega_1 \neq \omega_2$.

4.3 Power Spectra of Some Simple Systems

Let $\{X_n\}$ denote a zero-mean stationary random process with power spectral density $h(\omega)$.

1. THE AR(1) PROCESS:

Assume $\{X_n\}$ is an AR(1) process of the form $X_n = aX_{n-1} + \epsilon_n$ where $\{\epsilon_n\}$ is a purely random process with variance σ^2 .

Assume

$$\begin{aligned}X_n &= \int_{-\pi}^{\pi} e^{i\omega n} dZ_X(\omega) \\ \epsilon_n &= \int_{-\pi}^{\pi} e^{i\omega n} dZ_\epsilon(\omega)\end{aligned}$$

Substituting these forms into the updating equation for X_n gives

$$\int_{-\pi}^{\pi} e^{i\omega n} (1 - ae^{-i\omega}) dZ_X(\omega) = \int_{-\pi}^{\pi} e^{i\omega n} dZ_\epsilon(\omega)$$

For this equation to hold we require

$$(1 - ae^{-i\omega}) dZ_X(\omega) = dZ_\epsilon(\omega)$$

Multiplying each side of this equation by its complex conjugate and taking expectations gives

$$|1 - ae^{-i\omega}|^2 h(\omega) = h_{\epsilon\epsilon}(\omega)$$

and thus

$$h(\omega) = \frac{\sigma^2}{2\pi} |1 - ae^{-i\omega}|^{-2}$$

2. THE ARMA(k, l) PROCESS:

This process is defined by

$$X_n + a_1 X_{n-1} + \dots + a_k X_{n-k} = b_0 \epsilon_n + b_1 \epsilon_{n-1} + \dots + b_l \epsilon_{n-l} \quad (4.14)$$

where $\{\epsilon_n\}$ is a purely random process. Conditions for asymptotic stationarity are the same as those for stationarity of the AR(k) part of the model.

Using the spectral representation it is straightforward to show the power spectral density of the ARMA(k, l) process is

$$h(\omega) = \frac{\sigma^2 |b_0 + b_1 e^{-i\omega} + \dots + b_l e^{-i\omega l}|^2}{2\pi |1 + a_1 e^{-i\omega} + \dots + a_k e^{-i\omega k}|^2} \quad (4.15)$$

The flexibility of this form is one of the reasons the ARMA model has proved useful in practical applications. It is not used extensively in oceanography or atmospheric science, probably because it is not motivated by a relevant physical model.

3. SIMPLE LOW PASS FILTER:

$$Y_n = (X_{n+1} + X_{n-1})/2$$

This corresponds to a running mean of the $\{X_n\}$. Clearly $\{Y_n\}$ is a stationary random process. Using the spectral representation gives

$$dZ_Y(\omega) = \Gamma(\omega) dZ_X(\omega)$$

where

$$\Gamma(\omega) = (e^{i\omega} + e^{-i\omega})/2 = \cos(\omega)$$

Multiplying each side of this equation by its complex conjugate and taking expectations gives

$$h_{YY}(\omega) = |\Gamma|^2 h_{XX}(\omega) = \cos^2(\omega) h_{XX}(\omega)$$

Question: Why is this called a “low pass filter”? Plot assuming $\{X_n\}$ is a purely random process. Repeat the analysis for $Y_n = (X_{n+1} - X_{n-1})/2$. Why is this called a “high pass filter”?

4.4 Effective Degrees of Freedom

The concept of degrees of freedom, like the spectral representation, is difficult to understand at first but is worth the effort. It basically tells us how many useful bits of information are in a sample of autocorrelated observations. One of the surprising things is that the degrees of freedom depends on the quantity being estimated. We will use the concept when we look at the estimate power spectral density.

THE SAMPLE MEAN: Consider first the effective degrees of freedom for the sample mean \bar{X} , which we will assume is based on an average of N consecutive observations. It is straightforward to show that if $\{X_n\}$ is a weakly stationary random process, \bar{X} is an unbiased estimator of the mean of the process, and its asymptotic variance is (derive in class)

$$\text{var}(\bar{X}) \approx \frac{\sigma_X^2}{N} \sum_{s=-\infty}^{\infty} \rho(s)$$

If we had N^* independent observations the variance of the sample mean would be

$$\text{var}(\bar{X}) \approx \frac{\sigma_X^2}{N^*}$$

Equating these two expressions we obtain the effective degrees of freedom for the sample mean:

$$N^* = \frac{N}{\sum_{-\infty}^{\infty} \rho(s)}$$

If $\{X_n\}$ is an AR(1) process, the effective degrees of freedom for the sample mean is

$$N^* = N \frac{1 - a}{1 + a}$$

THE SAMPLE AUTOCOVARIANCE: Equation (3.6) gives the variance of the sample autocovariance for large sample size:

$$\text{var } \hat{\gamma}(k) \approx \frac{1}{N} \sum_{s=-\infty}^{\infty} \gamma^2(s) + \gamma(s+k)\gamma(s-k)$$

Consider the special case $k = 0$. We then have

$$\text{var}(s_X^2) \approx \frac{2\sigma_X^4}{N} \sum_{s=-\infty}^{\infty} \rho^2(s)$$

For a purely random process

$$\text{var}(s_X^2) \approx \frac{2\sigma_X^4}{N^*}$$

Comparing these two forms leads us to the “effective degrees of freedom” for the sample variance:

$$N^* = \frac{N}{\sum_{-\infty}^{\infty} \rho^2(s)}$$

where N is the sample size.

If $\{X_n\}$ is an AR(1) process then $\rho(k) = a^{|k|}$ and the effective degrees of freedom or the sample variance is

$$N^* = N \frac{1 - a^2}{1 + a^2}$$

Discuss limits an special cases.

4.5 Estimation of Power Spectral Density

Let $\{X_n\}$ denote a stationary random process with power spectral density denoted by $h(\omega)$. You might think that $|\zeta|^2$ is the “natural” estimator for h . (It is sometimes called the periodogram.) But there is a major problem! From the spectral representation

$$X = \int_{-\pi}^{\pi} e^{i\omega n} dZ(\omega)$$

it is straightforward to show

$$E(|\zeta(\omega_k)|^2) = \int_{-\pi}^{\pi} F_N(\omega - \omega_k) h(\omega) d\omega \quad (4.16)$$

As the record length increases, the Fejer kernel approaches a delta function and $|\zeta|^2$ approaches an unbiased estimator for h . So far, so good.

Let's now look at the variance of the estimator, $|\zeta|^2$. The finite Fourier transform of $2N + 1$ observations can be written $\zeta = \zeta_R + i\zeta_I$ where

$$\zeta_R(\omega_k) = \frac{\sqrt{\Delta\omega}}{2\pi} \sum_{n=-N}^N \cos(\omega_k n) X_n$$

and

$$\zeta_I(\omega_k) = -\frac{\sqrt{\Delta\omega}}{2\pi} \sum_{n=-N}^N \sin(\omega_k n) X_n$$

If $\{X_n\}$ is Gaussian, it is straightforward to show (I'll discuss in class)

$$(\zeta_R, \zeta_I) \sim N(0, \Sigma)$$

where Σ is a diagonal matrix with diagonal elements that can be approximated by $h/2$ for large sample sizes.

Noting that

$$|\zeta|^2 = \zeta_R^2 + \zeta_I^2$$

it follows that, for large sample sizes, we have the approximation

$$|\zeta|^2 \sim \frac{1}{2}h(\omega)\chi_2^2$$

The mean and variance of a χ_ν^2 random variable are ν and 2ν thus the variance of our estimator ($\nu = 2$, i.e. 2 degrees of freedom) is

$$\text{var}(|\zeta|^2) \approx h^2(\omega)$$

Here's the problem: The variance does not go to zero as $N \rightarrow \infty$.

The natural way around this problem is to evaluate $|\zeta(\omega_k)|^2$ for $k = -N, \dots, N$ and then smooth with a “spectral window”. In general, the wider the spectral window the greater the reduction in the variance of the spectral estimator (the “degrees of freedom” of the spectral estimator increases) but the “bias” increases. (The same trade off occurs when estimating a probability density function with a histogram.) I'll show some examples in class.

There are many spectral windows. (A Parzen window is used in my `crosspec.m` listed on the next two pages.) In my experience the width of the spectral window matters (shown as the blue horizontal line on the power spectral panel of `crosspec` output) but the shape of the window does not.

In practice chose the bandwidth by “window closing”: examining the stability of the estimated spectra while gradually closing the window (and hence reducing reliability). I'll show some examples in class.

Below I list a simple cross spectral program written in Matlab (crosspec.m) to smooth the periodogram using a Parzen spectral window. The width of the window is controlled by the input parameter, M . I usually take M to be about one tenth of the record length. Increasing M narrows the spectral window and vice versa. (M is the truncation point of the corresponding 'lag window'.)

```
function CROSS = crosspec(M,x,y,dt,plotit,oqlim);

% CROSS = crosspec(M,x,y,dt,plotit,oqlim);
% where on output CROSS = [freq Pxx Pyy K P Pxy].
% Replaces missing values (indicated by nan) by nanmean. Removes mean.
% Provides plot if nargin==5.
% Based on an M point Parzen window.
% See Priestley p444 for definition of Parzen window, p564 for plot.
% For Parzen window degrees of freedom equals 3.708614*N/M (p467)
% Bandwidth is 12/M, p527.
% oqlim is the upper frequency limit on the spectral plots.

I=find(isnan(x)==1); x(I)=nanmean(x); x=x-mean(x);
I=find(isnan(y)==1); y(I)=nanmean(y); y=y-mean(y);

n=length(x);
if rem(n,2)==1
    x=x(1:n-1);
    y=y(1:n-1);
end

x=x(:); y=y(:);

DX = fft(x);
DX = DX(:);
DY = fft(y); DY = DY(:);
n = length(DX);
delOmega = 2*pi/n; p = floor(2*n/M);
Omega = (-p:p)*delOmega +.000000000001;
Omega = Omega(:);
freq = [0:delOmega:pi];
freq = freq(:);
W = (sin(M*Omega/4)./sin(Omega/2)).^4.*(1-2/3*sin(Omega/2).^2).*6/(pi*M^3);
W = W/(sum(W)*delOmega);
W = W(:);
```

```

% compute the x power spectrum
INSTX = DX.*conj(DX);
INSTARTX = [INSTX(n-p+2:n);
INSTX(1:n/2+p+1)];
Pxx = delOmega*conv(INSTARTX,W)/(2*pi*length(x));
Pxx = Pxx(:);
Pxx = 2*Pxx(2*p:n/2+2*p);

% compute the y power spectrum
INSTY = DY.*conj(DY);
INSTARTY = [INSTY(n-p+2:n);
INSTY(1:n/2+p+1)];
Pyy = delOmega*conv(INSTARTY,W)/(2*pi*length(x));
Pyy = Pyy(:);
Pyy = 2*Pyy(2*p:n/2+2*p);

% compute the cross spectrum
INSTXY = DX.*conj(DY);
INSTARXY = [INSTXY(n-p+2:n);
INSTXY(1:n/2+p+1)];
Pxy = delOmega*conv(INSTARXY,W)/(2*pi*length(x));
Pxy = Pxy(:);
Pxy = 2*Pxy(2*p:n/2+2*p);

% compute the phase spectrum
P = atan2(-imag(Pxy),real(Pxy));
% compute the coherency spectrum
K = abs(Pxy)./sqrt(Pxx.*Pyy);

```