Everything adjoints

Today every vector space has an inner product.

Def The adjoint of a linear map $T: V \to W$ is the unique linear map $T^*: W \to V$ s.t. $\boxed{\langle Tv, w\rangle = \langle v, T^* w\rangle}$ for all $v \in V$ & $w \in W$.

Thm The adjoint exists.

Pf I'll add an appendix w/ the proof, but will skip it in class, as we won't have time to show the crucial lemma that proves it. (Riesz representation theorem.)

$\boxed{\text{Ex}}$ Let $T: \mathbb{C}^2 \to \mathbb{C}^3$ be given by $T(x,y) = (iy, x, 2x-y)$

We compute T^* from the definition:

$$\langle T(x,y), (a,b,c)\rangle = \langle (iy, x, 2x-y), (a,b,c)\rangle$$
$$= iy\bar{a} + x\bar{b} + (2x-y)\bar{c}$$
$$= x(\bar{b} + 2\bar{c}) + y(i\bar{a} - \bar{c})$$
$$= x(\overline{b+2c}) + y(\overline{-ia-c})$$
$$= \langle (x,y), (b+2c, -ia-c)\rangle$$

So $T^*(a,b,c) = (b+2c, -ia-c)$.

How about matrices?

$$[T] = \begin{bmatrix} 0 & i \\ 1 & 0 \\ 2 & -1 \end{bmatrix}$$

hmm...

$$[T^*] = \begin{bmatrix} 0 & 1 & 2 \\ -i & 0 & -1 \end{bmatrix}$$

$\cancel{(Tv)^\dagger w = v^\dagger T^\dagger w}$

Thm In any orthonormal basis, the matrix of T^* is the conjugate transpose of $[T]$. ie. $[T^*]_{ij} = \overline{[T]_{ji}}$

Pf Recall that the i-th coordinate of $v \in V$ in such a basis is given by $\langle v | e_i \rangle$.

So $[T^*]_{ij} = \langle e_j | T^* e_i \rangle$ (by definition)

$= \langle T e_j | e_i \rangle$ (adjointness)

$= \overline{\langle e_i | T e_j \rangle}$ (conjugate symmetry).

$= \overline{[T]_{ji}}$ / ∎

Def $T: V \to V$ is self-adjoint if $T^* = T$, ie. $\langle Tv, w \rangle = \langle v, Tw$

Prop T self adjoint w/ eigenvalue $\lambda \Rightarrow \lambda \in \mathbb{R}$.

Pf Let v be a corresponding eigenvector. Recall that $v \neq 0$, so $\langle v, v \rangle \neq$

Then $\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle$

$$\begin{aligned} \langle Tv, v \rangle &= \langle \lambda v, v \rangle = \lambda \langle v, v \rangle \\ \| \quad & \qquad\qquad\qquad\quad \| \\ \langle v, Tv \rangle &= \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle \end{aligned} \Rightarrow \boxed{\lambda = \bar{\lambda}} \text{ ie. } \lambda \text{ is r}$$

∎

RFx → RFx   LRFx

Spectral theorem for real operators

$\boxed{\text{Let } \mathbb{F} = \mathbb{R}.}$

In general, a real operator $V \xrightarrow{T} V$ ~~doesn't~~ might have no eigenvalues. But if T is self-adjoint:

Thm If T is self-adjoint, then it has a basis of eigenvectors, i.e. it's diagonalizable.

Ex Is

$$T = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 0 & 0 \\ 2 & 5 & 1 & 6 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 6 & 0 & 1 \end{array}$$

diagonalizable? Y, $T = T^T$ so self-adjoint.

Let's prove it:

Lemma Let $U \subseteq V$ be an invariant subspace by $T: V \to V$.

~~Then so is $U^\perp$.~~ Then $U^\perp$ is invariant under T^*.

Pf $v \in U^\perp$, we need to show that $T^* v \in U^\perp$.

$\hookrightarrow \langle u, v \rangle = 0 \ \forall u \in U \Rightarrow \langle Tu, v \rangle = 0 \ \forall u \in U$, b/c $Tu \in U$ (invariance)

$\Rightarrow \langle u, T^* v \rangle = 0$ (adjointness) $\forall u \in U$

$\boxtimes$

Cor $\underbrace{T = T^*, U \subseteq V}_{T\text{-invariant}} \Rightarrow \underbrace{U^\perp}_{T\text{-invariant.}}$

Lemma $T: V \to V$ self-adjoint. Then it has an eigenvalue.

Pf It suffices to study the case $T: \mathbb{R}^n \to \mathbb{R}^n$. Consider $A = [T]$. Treated as a matrix w/ complex entries, it defines a transformation $A: \mathbb{C}^n \to \mathbb{C}^n$, which must have an eigenvector $z \in \mathbb{C}^n$ w/ eigenvalue $\lambda \in \mathbb{C}$. But bc T is self adjoint, $\lambda \in \mathbb{R}$.

Moreover, write $z = u + iv$, where $u, v \in \mathbb{R}^n$. Then

$$\underbrace{Tu + iTv =: T(u+iv) = \lambda(u+iv) = \lambda u + i\lambda v.}$$

$$\begin{cases} Tu = \lambda u \\ Tv = \lambda v \end{cases}$$

Note that $z \neq 0$ (eigenvector) $\Rightarrow$ either $u \neq 0$ or $v \neq 0$ (or both). So at least one of them will give an eigenvector for T. ☒

Rmk this works for complex self adjoint operators.

Pf of the spectral theorem

Let $T: V \to V$ be a self adjoint operator, and $v \in V$ an eigenvector w/ eigenval $\lambda \in \mathbb{R}$. Then $U = \operatorname{span}(v)$ is invariant $\Rightarrow U^\perp$ is invariant, and $(n-1)$ dimen. Then $T: U^\perp \to U^\perp$ is self adjoint. Repeat $(n-1)$-times. Done.

☒