Isometries: unitary & orthogonal operators

An isometry is a linear map that preserves distances.

Def $T: V \to W$ is an isometry if $\|Tv\| = \|v\|$ for all $v \in V$.

e.g. $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

non e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Prop TFAE

$T: V \to W$

$\overset{\uparrow}{e_1, \dots, e_n}$ $\quad$ $\overset{\uparrow}{f_1, \dots, f_m}$

orthonormal

- ⓐ T is an isometry
- ~~ⓑ $T^{*}T = I$~~ ⓑ $\langle Tu, Tv\rangle = \langle u, v\rangle$
- ~~ⓒ $\langle Tu, Tv\rangle = \langle u, v\rangle$~~ ⓒ $T^{*}T = I$
- ⓓ $Te_1, \dots, Te_n$ is orthonormal
- ⓔ the columns of $[T]$ are orthonormal

Pf (a)⇒(b) is the hardest, because it uses a polarization identity, which we skipped: In the real case, it says that

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

So

$$\langle Ux, Uy\rangle = \frac{1}{4}\left(\|Ux+Uy\|^2 - \|Ux-Uy\|^2\right)$$

$$= \frac{1}{4}\left(\|U(x+y)\|^2 - \|U(x-y)\|^2 = \frac{1}{4}(\|x+y\|^2 + \|x-y\|^2\right.$$

$$= \langle x, y\rangle$$

$=$

(b)⇒(c) $\langle x, y \rangle = \langle Ux, Uy \rangle = \langle x, U^* U y \rangle$ for all x, y.

So $y = U^* U y$. (~~$\langle x, - \rangle : V \to \mathbb{F}$ is nonzero~~

(b)⇒(d) $\begin{cases} \langle e_i, e_j \rangle = 0 \\ \|e_i\| = 1 \end{cases} \longrightarrow \begin{cases} \langle Te_i, Te_j \rangle = 0 \\ \forall \|Te_i\| = 1 \end{cases}$ (b)

(d)⇒(e) immediate from (d).

☒

Ex Go back and check that the matrices in the previous page are or aren't isometries.

Def An operator $T: V \to V$ is

- **Unitary** if $\mathbb{F} = \mathbb{C}$, T is invertible & an isometry.

- **orthogonal** if $\mathbb{F} = \mathbb{C}$, T is invertible & an isometry.

Prop TFAE
(a) T is unitary
(b) T is an isometry
(c) $T^*T = TT^* = I$
(d) $T^{-1} = T^*$
(e) $Te_1, \dots, Te_n$ is an orthonormal basis for V.
(f) the rows form a orthonormal basis
(g) T^* is unitary.

Pf Whoop! This is straightforward.

(b)$\Rightarrow$(a) bc $Tv = 0 \Rightarrow \|Tv\| = \|0\| \Rightarrow v = 0$, so T is
$$\overset{\|}{\|v\|}$$
injective $\Rightarrow$ invertible.

(a)$\Rightarrow$(c)/(d) isometry $\Leftrightarrow$ $S^*S = I \Rightarrow S^*\cancel{SS}^{-1} = S^{-1} \rightarrow \boxed{S^* = S^{-1}}$
unitary $\rightarrow$ invertible $\rightarrow$ $S^{-1}S = I = SS^{-1}$

d$\Rightarrow$(e) Follows from our characterization of invertibility.

(c)$\Rightarrow$(g) Clearly.

Now (g)$\Leftrightarrow$(f). bc the rows of T are almost the columns of T^* — but they don't change orthogonality or norm.

$\boxtimes$

(Finish the implications.

Break

Prop Suppose $T: V \to V$ is an isometry, or self-adjoint and $\lambda_1 \neq \lambda_2$ are eigenvalues w/ eigenvectors v_1 & v_2. Then $\langle v_1, v_2\rangle = 0$.

Pf If isometry: ~~WLOG, $\lambda_2 \neq 0 \neq \lambda_1$~~.

- if $\lambda_1 = 0$, then $\langle v_1, v_2\rangle = \langle Tv_1, Tv_2\rangle = \langle 0, \lambda_2 v_2\rangle = 0$

- if $\lambda_1 \neq 0 \neq \lambda_2$, then $\langle v_1, v_2\rangle = \langle Tv_1, Tv_2\rangle = \underbrace{\lambda_1 \bar{\lambda}_2}_{\neq 0} \langle v_1, v_2\rangle$

This only happens if $\langle v_1, v_2\rangle = 0$.

If self-adjoint: $\lambda_1 \neq 0$:

$$\langle Tv_1, v_2\rangle = \langle \lambda_1 v_1, v_2\rangle = \lambda_1 \langle v_1, v_2\rangle$$
$$\parallel \qquad\qquad\qquad\qquad \parallel$$
$$\langle v_1, T^{*} v_2\rangle = \langle v_1, \bar{\lambda}_2 v_2\rangle = \lambda_2 \langle v_1, v_2\rangle$$

only if $\langle v_1, v_2\rangle = 0$.

Cor If T is diagonalizable, then it has a basis of orthonormal eigenvectors.

Pf Find an orthonormal basis on each eigenspace.

This means that if T is diagonalizable and self-adjoint OR unitary/isometry, then ~~E the~~ we can write $[T] = U D U^{-1}$, where U is orthogonal/~~isometry~~ unitary.

(We will do an example at the end of the notes.)

But for $\mathbb{F}=\mathbb{R}$, an orthogonal matrix might not be diagonalizable! e.g $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$\mathbb{F}=\mathbb{C}$.

Def $N: V \to V$ is normal if $N^*N = N N^*$.

e.g self-adjoint $\Rightarrow$ normal.

e.g. unitary $\Rightarrow$ normal.

Thm (Spectral thm for normal ops). Let N be a normal operator on complex vector space. Then N is diagonalizable

Pf Take an orthonormal basis where $A=[N]$ is upper triangular. ~~Literally compare the entries of AA^* and A^*A. e.g $[AA^*]_{11} =$~~ $\longrightarrow$

Then $\|Te_1\|^2 = |A_{1,1}|^2$.

$\parallel$

and $\|T^* e_1\|^2 = |A_{1,1}|^2 + \dots + |A_{1,n}|^2$

so $|A_{1,2}|^2 + \dots + |A_{1,n}|^2 = 0 \Rightarrow A_{1,2}, \dots, A_{1,n} = 0$.

Now just repeat on each column. $\square$

Example: find an orthonormal basis of eigenvectors for $S: \mathbb{C}^2 \to \mathbb{C}^2$ $S(x,y) = (x-3y, 3x+y)$.

On the exam you probably found

$v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \leftrightarrow \lambda_1 = 1+3i$

$U = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{10}}$

$[S] = U \begin{bmatrix} 1+3i & 0 \\ 0 & 1-3i \end{bmatrix} U^{-1}$

$v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \leftrightarrow \lambda_2 = 1-3i$

These are ~~not~~ orthogonal bc the eigenvals are different. They are not normalized yet:

$\|v_1\| = \sqrt{1^2 + 3^2} = \sqrt{10} = \|v_2\|$

So orthonormal basis of eigenvectors is $\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$ & $\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ i \end{bmatrix}$.

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Example find a basis of eigenvectors for $\begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}$

sol: $\frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$