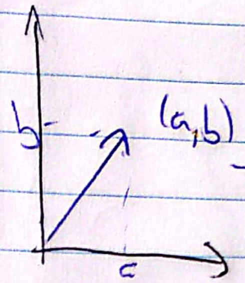
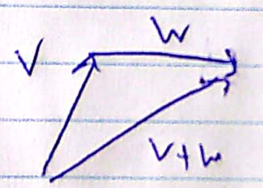
When we have a vector space V, there is no canonical notion of "length" of a vector, or the "angle" between two vectors. This differ from the situation in $\mathbb{R}^n$



$(a,b) \longrightarrow \|v\| = \sqrt{a^2+b^2}$

$\longrightarrow$ in general, $\|v\| = \sqrt{a_1^2 + \dots + a_n^2}$

Length is a number $\|v\| \in \mathbb{R}$, but the function $\mathbb{R}^n \xrightarrow{\|\cdot\|} \mathbb{R}$ is not linear (e.g. it doesn't respect sums).



$\|v\| + \|w\| \geq \|v+w\|$

We introduce the dot product to be more linear:

$v \cdot w = a_1b_1 + \dots + a_nb_n$

$v \to (a_1,\dots,a_n)$

$w \to (b_1,\dots,b_n)$

Note that $v \cdot v = \|v\|^2$.

This definition is very much basis dependent, though. Like if you pick another nice basis, like an eigenbasis, then, $v \cdot w \neq a_1b_1 + \dots + a_nb_n$.

Also, it doesn't help a lot with the abstract case (what's a dot product of polynomials?). We could do a basis dependent approach, but that's not good way.

$\langle , \rangle : V \times V \to \mathbb{R}$.

Def An inner product on a real vector space V is a function taking a pair of vectors (v, w) to a real number $\langle v, w\rangle \in \mathbb{R}$ s.t.

ⓐ $\langle v, v\rangle \geq 0$ positivity.
ⓑ $\langle v, v\rangle = 0 \iff v = 0$ nondegenerate
ⓒ $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ } bilinearity
ⓓ $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$
ⓔ $\langle u, v\rangle = \langle w, v\rangle$ symmetry

An inner product defines a norm by $\|v\| := \langle v, v\rangle$

What about $\mathbb{C}^n$?

$$\|\vec{z}\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$
$$= \sqrt{\cancel{\phi} z_1 \overline{z_1} + \cdots + z_n \overline{z_n}}$$

Suggests

$$\vec{z} \cdot \vec{w} = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$$

but then

$$\vec{w} \cdot \vec{z} = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}$$

Def Complex inner product: ⓐ, ⓑ, ⓒ, ⓓ but
$$\langle v, w\rangle = \overline{\langle w, v\rangle}$$

Prop ① $\langle -, v\rangle$ is linear

② $\langle 0, v\rangle = 0 \quad \forall v \in V$

③ $\langle v, 0\rangle = 0 \quad \forall v \in V$

④ $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle \quad \forall u,v,w \in V$

⑤ $\langle u, \lambda v\rangle = \bar{\lambda}\langle u,v\rangle \quad \forall \lambda \in \mathbb{C},\ u,v \in V$

—||—

Def $\|v\| = \sqrt{\langle v,v\rangle}$ → e.g.

Prop $\|v\| = 0 \Leftrightarrow v = 0$

$\|\lambda v\| = |\lambda|\,\|v\|$.

—|—

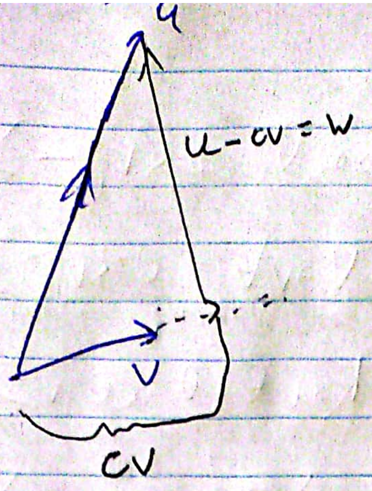
BREAK

→ e.g. $\langle u,v\rangle = \|u\|\,\|v\|\cos\theta$

Def $u, v \in V$ are orthogonal $\langle u,v\rangle = 0$

Prop 0 is the only vector orthogonal to all vectors including itself

Prop $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if $u \perp v$



$$0 = \langle u - cv, v \rangle = \langle u,v \rangle - c\|v\|^2 \to c = \frac{\langle u,v \rangle}{\|v\|^2}$$

$u = cv + w,\ v \perp w.$

Prop [Cauchy-Schwarz] $|\langle u,v \rangle| \leq \|u\| \, \|v\|$

Pf $u = \frac{\langle u,v \rangle}{\|v\|^2} v + w \longrightarrow \|u\|^2 = \left\| \frac{\langle u,v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$

$u = \frac{\langle u,v \rangle}{\|v\|^2} v$ $\qquad$ $\cancel{0 = \frac{\langle u,v \rangle}{\|u\|}}$ $\qquad$ $\Leftrightarrow w = 0$ $\qquad$ $= \frac{|\langle u,v \rangle|^2}{\|v\|^4} \|v\|^2 + \|w\|^2$

$\geq \frac{|\langle u,v \rangle|^2}{\|v\|^2} + \|w\|^2$

multiply $\|v\|^2$, $\triangle$

fin.

Prop (triangle inequality) $\|u+v\| \leq \|u\| + \|v\|$

$\|u+v\| = \langle u+v, u+v \rangle$

$= \langle u,v \rangle + \langle v,u \rangle + \langle u,u \rangle + \langle v,v \rangle$

$= 2\mathrm{Re}\langle u,v \rangle + \|u\|^2 + \|v\|^2$

$\leq 2\langle u,v \rangle + \|u\|^2 + \|v\|^2$

$\leq 2\|u\| \, \|v\| + \|u\|^2 + \|v\|^2 = (\|u\| + \|v\|)^2$