Reminders

Def An <u>inner product space</u> is a vector space V over $\mathbb{F}$ equipped with an operation $\langle , \rangle$ taking two vectors u, v and giving a <u>number</u> $\langle u,v \rangle \in \mathbb{F}$, such that

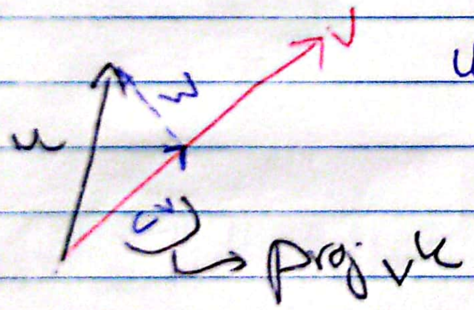
① $\langle \alpha v + w, u \rangle = \alpha \langle v,u \rangle + \langle w,u \rangle$

② $\langle v,w \rangle = \overline{\langle w,v \rangle}$

③ $\langle v,v \rangle \geq 0$; $\langle v,v \rangle = 0 \iff v = 0$

$\rightarrow \langle u,v \rangle = 0$

Rmk (Orthogonal decomposition)



$u = cv + w$

↳ $\langle v,w \rangle = 0$

$$c = \frac{\langle u,v \rangle}{\|v\|^2}$$

Prop (Pythagoras) $\langle u,v \rangle = 0 \Rightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2$

Cor $\langle u,v \rangle = 0 \Rightarrow u, v$ are l.i.

Pf $\|au + bv\|^2 = \|au\|^2 + \|bv\|^2$

$= |a| \|u\|^2 + |b| \|v\|^2 > 0 \Rightarrow au + bv \neq 0$

Prop (Cauchy-Schwarz) $|\langle u,v\rangle| \leq \|u\|\,\|v\|$

Pf We saw that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ iff $\langle u,v\rangle = 0$ (Pythagoras). Consider the orthogonal decomposition

$$u = \frac{\langle u,v\rangle}{\|v\|^2} v + w$$

Since $\langle v,w\rangle = 0$, by Pythagoras:

$$\|u\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$$

$$\|u\|^2 = \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \|w\|^2 \Rightarrow \|u\|^2 \|v\|^2 = |\langle u,v\rangle|^2 + \underbrace{\|w\|}_{\geq 0}$$

∎

Cor $\|u+v\| \leq \|u\| + \|v\|$

Pf $\|u+v\|^2 = \langle u+v, u+v\rangle^2$

$$\overline{\langle u,v\rangle}$$

$$= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle$$

$$= \|u\|^2 + \|v\|^2 + 2\mathrm{Re}(\langle u,v\rangle)$$

$$\leq \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle|^2$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\| = (\|u\| + \|v\|)^2$$

∎

Def A vector $v \in V$ is normalized if $\|v\| = 1$

→ You can "normalize" a vector by replacing it with $\frac{v}{\|v\|}$; → $\left\| \frac{v}{\|v\|} \right\| = \frac{\|v\|}{\|v\|} = 1$

e.g. the vector $(1,1) \in \mathbb{R}^2$ has norm $\sqrt{1^2+1^2} = \sqrt{2}$, so it's normalization is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Def An orthonormal list is $e_1, \dots, e_n \in V$ which is normalized and mutually orthogonal.

$$\hookrightarrow \begin{cases} \|e_i\| = 1 \\ \langle e_i, e_j \rangle = 0 \end{cases}$$

e.g. - (Canonical basis in $\mathbb{F}^n$)

- $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

- $1, \frac{\sin(\pi x)}{\sqrt{2}}, \frac{\sin(2\pi x)}{\sqrt{2}}, \dots, \frac{\cos(\pi x)}{\sqrt{2}}, \frac{\cos(2\pi x)}{\sqrt{2}}, \dots$

- NON EXAMPLE: $1, x, x^2, x^3 \in \mathcal{P}_3(\mathbb{R})$.

Prop $\|a_1e_1+\dots+a_ne_n\|^2 = |a_1|^2+\dots+|a_n|^2$
for an orthonormal list

Cor Every orthonormal list is l.i.

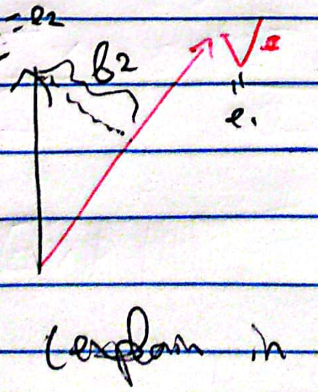
Pf $a_1e_1+\dots+a_ne_n=0 \Rightarrow |a_1|^2+\dots+|a_n|^2=0$
$\Rightarrow a_1=\dots=a_n=0.$ ☒

Def An <u>orthonormal basis</u> is an orthonormal list of length dim V.

~~GRAM~~-
ORTHONORMALIZATION (aka Gram-Schmidt)
OF A BASIS

Start w/a basis $e_1,\dots,e_n$. We want to produce an orthonormal basis out of it.

e.g.



$f_1 = \dfrac{v}{\|v\|}$ (normalized)

$\hat{f}_2 = u - \mathrm{proj}_u^{v} \to f_2 = \dfrac{\hat{f}_2}{\|\hat{f}_2\|}$

etc. (explain in class)