Midterm 2

Matrix Theory & Linear Algebra II

Your name: **Banner ID:**

Assume that you have to explain your reasoning even if the question doesn't explicitly asks you to.

1. (40 points) Consider the linear transformation $S : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$S(x, y) = (x - 3y, 3x + y).$$

   (a) Find a basis of eigenvectors for S. *Hint: start by writing down the matrix of S.*

   (b) Write down the (diagonal) matrix of S in the basis you found in (a).

Solution.

The matrix of S in the canonical basis is $[S] = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$. The characteristic polynomial is

$$\det\left(\begin{bmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 + 9 = \lambda^2 - 2\lambda + 10.$$

The roots of are given by the quadratic equation:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \implies \lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 10}}{2} = 1 \pm 3i.$$

So the eigenvalues are $1 + 3i$ and $1 - 3i$.

- $\lambda = 1 + 3i$: we need to solve the equation

$$Sv = (1 + 3i)v \iff (x - 3y, 3x + y) = (x + 3ix, y + 3iy).$$

  The first entry tells us that $y = -ix$. Hence an eigenvector is $v_1 = (1, -i)$.

- $\lambda = 1 - 3i$: we need to solve the equation

$$Sv = (1 - 3i)v \iff (x - 3y, 3x + y) = (x - 3ix, y - 3iy).$$

  The first entry tells us that $y = ix$. Hence an eigenvector is $v_2 = (1, i)$.

The two eigenvectors are related to distinct eigenvalues, so they are linearly independent and hence form a basis.

In this eigenbasis, the matrix is diagonal, with the eigenvalues as in the diagonal:

$$[S]_{v_1, v_2} = \begin{bmatrix} 1 + 3i & 0 \\ 0 & 1 - 3i \end{bmatrix}.$$

■

2. (30 points) *Hint: you don't have to calculate anything in this question.*

Let V be a 5-dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$. Assume that, with respect to some basis $\mathcal{B} = e_1, e_2, e_3, e_4, e_5$ of V, we have:

$$[T]_{\mathcal{B}} = \begin{bmatrix} 2+i & 3 & -1 & 4 & 7 \\ 0 & 1 & 5 & -6 & -10 \\ 0 & 0 & -2i & 2 & 2i \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

(a) Find the eigenvalues of T.

(b) Is T diagonalizable?

Solution. Since the matrix is upper triangular, the eigenvalues are the diagonal entries: $2+i$, 1, $-2i$, 0, and 3. Since these are 5 distinct eigenvalues in a 5-dimensional vector space, the 5 corresponding eigenvectors will be linearly independent, and hence form a basis. So the matrix matrix is diagonalizable. ■

3. (30 points) Let $\mathcal{P}_2(\mathbb{R})$ denote the vector space of real polynomials with degree at most 2, and consider the linear transformation $D : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ defined by $Dp = p'(x)$.

(a) Find all eigenvalues and eigenvectors of D.

(b) Is D diagonalizable?

Solution. First solution: for $p(x) = a + bx + cx^2$, we have $(Dp)(x) = b + 2cx$. The eigenvalue problem is $Dp = \lambda p$. If $c \neq 0$, then the RHS is a degree 2 polynomial while the LHS has degree 1. So if $c \neq 0$ then p is not an eigenvector. Similarly, if $b \neq 0$ then p is not an eigenvector.

It remains to analyze the constant polynomials, i.e. $p(x) = a$. Then $Dp = 0 = 0 \cdot a$. So constant polynomials are eigenvectors, associated to the zero eigenvalue.

The transformation is not diagonalizable, since the space is 3 dimensional and the eigenvectors only span a 1 dimensional vector space - so we can't find a basis of eigenvectors.

Second solution: the matrix of D in the basis $\{1, x, x^2\}$ is $[D] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The characteristic polynomial is

$$\det([D - \lambda I]) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = -\lambda^3.$$

The only root is $\lambda = 0$, which is then the unique eigenvalue. The eigenvectors are solutions to the problem

$$Dp = 0 \iff b + 2cx = 0 \implies b = c = 0.$$

So the eigenvectors are the nonzero polynomials with $b = c = 0$, i.e. the constants. ■