Orthogonal complements & projections

Matrix Theory & Linear Algebra II

Definition 1. The **orthogonal complement** of a subspace $U \subseteq V$ is the subset $U^\perp$ of all vectors orthogonal to vectors in U:

$$U^\perp = \{v \in V : \langle u, v \rangle = 0 \text{ for all } u \in U\}.$$

Exercise 2. Use the axioms of inner products to check that $U^\perp$ is also a subspace of V.

Exercise 3. Convince yourself that the orthogonal a plane in $\mathbb{R}^3$ containing the origin is the line through the origin perpendicular to that plane.

The following is crucial:

Proposition 4. *Let $U \subseteq V$ be a subspace. Given a vector $v \in V$, there exist unique vectors $u \in U$ and $w \in U^\perp$ such that $v = u + w$.*

Proof. Take an orthonormal basis $e_1, \ldots, e_m$ of U, and extend it to an orthonormal basis of V by adding vectors $f_1, \ldots, f_n$.[1] We will show that $f_1, \ldots, f_n$ is a basis for $U^\perp$, in which case the decomposition

$$v = \underbrace{a_1 e_1 + \cdots + a_n e_n}_{u \in U} + \underbrace{b_1 f_1 + \cdots + b_m f_m}_{w \in U^\perp}$$

is the desired (unique decomposition).

Well, since $e_1, \ldots, e_m, f_1, \ldots, f_n$ is an orthonormal basis of V. In particular, this means that for any $u \in U$ we have

$$\begin{aligned}
\langle u, f_i \rangle &= \langle a_1 e_1 + \cdots + a_n e_n, f_i \rangle \\
&= \langle a_1 e_1, f_i \rangle + \cdots + \langle a_m e_m, f_i \rangle \\
&= a_1 \langle a_1 e_1, f_i \rangle + \cdots + a_m \langle e_m, f_i \rangle \\
&= 0,
\end{aligned}$$

where in the last line we use that the basis is orthogonal. But this means, by definition, that $f_i \in U^\perp$. That's what we wanted. $\square$

Definition 5. Let $U \subseteq V$ be a subspace. The *orthogonal projection* is the operator $P_U : V \to V$ defined as follows: Given a vector $v \in V$, use the previous lemma to decompose it as $v = u + w$, where $u \in U$ and $v \in U^\perp$.Then $P_U(v) = u$.

Exercise 6. Use the axioms of inner products to check that P_U is indeed a linear transformation.

Lemma 7. *range* $P_U = U$ *and* $\ker P_U = U^\perp$

[1] We can do these steps because of Gram-Schmidt.

Proof. The image of P_U is clearly contained in U, and it is the identity on U, so the image is U itself.

If $P_U(v) = 0$, then $u = 0$ in the decomposition $v = u + w$, where $u \in U$ and $w \in U^\perp$. But then $v \in U^\perp$. $\square$

Exercise 8. Conclude that $\dim U^\perp = \dim V - \dim U$. Compare this with Exercise 3.

Corollary 9. *P_U is self adjoint, i.e. $\langle P_U(v), v'\rangle = \langle v, P_U(v')\rangle$ for all $v, v' \in V$.*

Proof. Decompose $v = u + w$ and $v' = u' + w'$, where $u, u \in U$ and $w, w' \in U^\perp$. On one hand

$$\begin{aligned}\langle P_U(v), v'\rangle &= \langle u, u' + w'\rangle \\ &= \langle u, u'\rangle + \langle u, w'\rangle. \\ &= \langle u, u'\rangle.\end{aligned}$$

On the other,

$$\begin{aligned}\langle u, P_U(v')\rangle &= \langle u + w, u'\rangle \\ &= \langle u, u'\rangle + \langle w, u'\rangle \\ &= \langle u, u'\rangle.\end{aligned}$$

The expressions are the same! $\square$

Given a vector $v \in V$ and a subspace $U \subseteq U$, here is an *extremely natural question*: what's the smallest distance between v and U? That is, what is the vector

Our intuition in 3d tells that this minimum is achieved by studying the decomposition $v = u + w$, where $u \in U$ and $w \in U^\perp$. The minimum should be $\|w\| = \|v - u\|$. Indeed:

Proposition 10. *Let $v \in V$ and $v = u + w$, where $u \in U$ and $w \in U^\perp$. Let $u' \in U$ be another vector in U. Then $\|v - u\| \leq \|v - u'\|$.*

Proof.

$$\begin{aligned}\|v - u\|^2 &\leq \|v - u\|^2 + \|u - u'\|^2 \\ &\leq \|v - u + u - u'\|^2 \\ &= \|v - u'\|,\end{aligned}$$

where we used Pythagoras (why are $v - u$ and $u - u'$ orthogonal?). $\square$

So to find minimize the distance between v and U we calculate $P_U(v)$. If we pick an orthonormal basis $\mathcal{B} = e_1, \ldots, e_n$ for U, we have a formula for that[2]

$$P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n = \begin{bmatrix}\langle v, e_1\rangle \\ \vdots \\ \langle v, e_n\rangle\end{bmatrix}_{\mathcal{B}}.$$

The nice thing about this formula is that it typechecks even when V is infinite dimensional. For instance let $V = C^0([0, 1])$, the vector space of continuous functions, and consider the subspace space spanned by the functions 1, $\sin(x)$, $\cos(x)$:

$$U = \text{span}(1, \sin(x), \cos(x)).$$

Then the projection of any other function $f(x) \in V$ is given by

$$P_U(f)(x) = \int_0^1 f(x)dx + \sin x \int_0^1 f(x)\sin x dx + \cos x \int_0^1 f(x)\cos x dx.$$

The coefficients above are just inner products. This kind of reasoning explains the *Fourier transform* which you might study later on. This is very important for electronics, etc.

[2] To convince yourself of this, pick a basis of U, and extend it to V.