Matrix Theory & Linear Algebra II

Practice Midterm I

Your name: **Banner ID:**

1. (40 points) In this question you are presented with three statements. *Every one of them is wrong.* For each one, give a short (one or two sentences) explanation of why.

   (a) Complex conjugation, defined by $\overline{a+bi} = a - bi$, is a linear transformation $\mathbb{C} \to \mathbb{C}$.

   (b) The set 2×2 invertible real matrices
   $$GL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$
   is a subspace of $M_{2,2}(\mathbb{R})$.

   (c) If $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are linearly independent sets of vectors in a vector space V, then $\{v_1 + w_1, v_2 + v_2, v_3 + w_3\}$ is also linearly independent.

2. (30 points) For each of the following questions, draw a small sketch representing the corresponding operation.

   (a) Find the matrix for the linear transformation that reflects every vector in $\mathbb{R}^2$ about the line $y = x$.

   (b) Find the matrix for the linear transformation that stretches $\mathbb{R}^2$ by a factor of 4 in the vertical direction.

   (c) Find the matrix for the linear transformation that performs the operation described in (b), followed by the operation described in (a).

3. (30 points) An **eigenvector** for a linear operator $T : V \to V$ is a vector $v \in V$ such that $T(v) = \lambda \cdot v$ for some scalar $\lambda \in \mathbb{F}$. An **eigenbasis** for T is a basis in which every vector is an eigenvector.

   Let $\mathcal{P}_3(\mathbb{C})$ denote the complex vector space of complex polynomials, and denote by $U : \mathcal{P}_2(\mathbb{C}) \to \mathcal{P}_2(\mathbb{C})$ the linear transformation defined by
   $$U(a + bx + cx^2) = (a - b) + (a + b)x.$$

   (a) The set $\{i + x, -i + x, x^2\}$ is a basis for $\mathcal{P}_2(\mathbb{C})$. Show that this basis is in fact an eigenbasis for U, and write down the matrix for U in this basis.
   Hint: $i - 1 = i(i + 1)$ and $i + 1 = -i \cdot (i - 1)$.

   (b) What is the dimension of the null space of U? What is the dimension of its range?