Problem Set 4

Matrix Theory & Linear Algebra II

(1) Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T.$$

(2) Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that

$$(ST)^{-1} = T^{-1}S^{-1}$$

Solution. The exercise is asking to show that the inverse of ST is $T^{-1}S^{-1}$. Indeed,

$$T^{-1}S^{-1}ST = T^{-1}T = I_U.$$

and

$$STT^{-1}S^{-1} = SS^{-1} = I_V.$$

■

(3) Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

Solution. It's easier to construct invertible linear maps than to use dimension arguments. (This problem is still a little wordy.)

On one hand, define a map $\phi : V \to \mathcal{L}(\mathbb{F}, V)$ sending a vector $v \in V$ to the linear map $\phi_v \in \mathcal{L}(\mathbb{F}, V)$ defined by $\phi_v(k) = k \cdot v$. On the other, define a map $\psi : \mathcal{L}(\mathbb{F}, V) \to V$ sending a linear map $f : \mathcal{L}(\mathbb{F}, V)$ to $f(1) \in V$.

Then $\psi \circ \phi : V \to V$ is the map sending $v \in V$ to $\phi_v(1) = 1 \cdot v = v$ by linearity, so it's the identity map. Additionally, $\phi \circ \psi : \mathcal{L}(\mathbb{F}, V) \to \mathcal{L}(\mathbb{F}, V)$ is the map sending a linear map $f \in \mathcal{L}(\mathbb{F}, V)$ to the linear map $\phi_{f(1)} \in \mathcal{L}(\mathbb{F}, V)$, which is defined by $\phi_{f(1)}(k) = k \cdot f(1) = f(k)$ by linearity. So $\phi_{f(1)} = f$ and $\phi \circ \psi$ is an identity map. ■

(4) Show that $M_{n \times n}(\mathbb{F})$ and $\mathbb{F}^{n^2}$ are isomorphic vector spaces.[1]

Solution. Both vector spaces have dimension n^2, so they are isomorphic. *Bonus:* construct an isomorphism. For instance, see the proof in the book that vector spaces with the same dimension are isomorphic. ■

(5) Show that $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic are *real* vector spaces.

[1] There was no question in the posted problem set.

Proof. Define a map $T : \mathbb{C} \to \mathbb{R}^2$ defined by $T(a+bi) = \left[a//b\right]$. Check that it is linear, and that it has no kernel. Then it is an isomorphism, because both spaces have the same dimension. $\square$

(6) True or false:

(a) Every linear operator in an n-dimensional vector space has n distinct eigenvalues;
(b) If a matrix has one eigenvector, it has infinitely many eigenvectors;
(c) There exists a square real matrix with no real eigenvalues;
(d) There exists a square matrix with no (complex) eigenvectors;
(e) Similar matrices always have the same eigenvalues;
(f) Similar matrices always have the same eigenvectors;
(g) A non-zero sum of two eigenvectors of a matrix A is always an eigenvector;
(h) A non-zero sum of two eigenvectors of a matrix A corresponding to the same eigenvalue λ is always an eigenvector.

Solution. To each *false* assertion, there are *several* counterexamples. I'll give only one for each. Yours might be very different.

(a) F, the identity transformation on $\mathbb{F}^n$ has only one eigenvalue.
(b) T, any scalar multiple of an eigenvector v is also an eigenvector.
(c) T, any (non-trivial) rotation matrix.
(d) F, any complex linear transformation has an eigenvalue.
(e) T, similar matrices represent the same linear transformation in different bases. You can also argue that the determinant preserves products of matrices.
(f) F, if $M = S^{-1}AS$ and v is an eigenvector of A, then $S^{-1}(v)$ is an eigenvector of T, but not v.
(g) F, for instance if the eigenvalues are distinct then the vectors are l.i., so their sum is in neither eigenspace, so their sum might not be an eigenvector.
(h) T, if v is an eigenvector with eigenvalue λ then $T(c \cdot v) = c \cdot T(v) = c\lambda \cdot v = \lambda \cdot (cv)$.

$\blacksquare$

(7) Compute the eigenvalues and eigenvectors of the rotation matrix

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Note that the eigenvalues (and eigenvectors) do not need to be real.

Proof. The determinant of $A - \lambda I$ is

$$\det(A-\lambda I) = (\cos\alpha-\lambda)^2+(\sin\alpha)^2 = 1-2\cos\alpha\cdot\lambda+\lambda^2 = (\lambda+\sqrt{\cos\alpha-1})+(\lambda+\sqrt{\cos\alpha-1}).$$

So the solutions to $\det(A-\lambda I) = 0$ are $\sqrt{\cos\alpha - 1})$ and $\sqrt{\cos\alpha+1})$. The eigenvectors... $\square$

(8) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis in a vector space V. Assume also that the first k vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of the basis are eigenvectors of an operator A, corresponding to an eigenvalue λ (i.e. that $A\mathbf{v}_j = \lambda\mathbf{v}_j, j = 1, 2, \ldots, k$). Show that in this basis the matrix of the operator A has block triangular form

$$\begin{pmatrix} \lambda I_k & * \\ 0 & B \end{pmatrix},$$

where I_k is a $k \times k$ identity matrix and B is some $(n-k) \times (n-k)$ matrix.

Solution. The j-th column of A is $A(v_j)$, so for $1 \leq j \leq k$ we have $A(v_j) = \lambda_j$, i.e. the only non-zero entry is in the j-th row. The other columns are just generic columns. ■

(9) An operator A is called nilpotent if $A^k = 0$ for some k. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

Proof. There are two claims: that 0 is an eigenvalue, and that it is the only eigenvalue.

We first prove the latter. Suppose that $A^k = 0$. If v is an eigenvector with eigenvalue λ, then $A^k(v) = \lambda^k v$, on the other hand $A^k(v) = 0$. So $\lambda^k = 0 \implies \lambda = 0$.

Now we show that 0 is an eigenvalue. Let $v \in V$ be any *nonzero* vector. Let ℓ be the smallest number such that $A^\ell(v) = 0$. Note that $1 \leq \ell \leq k$. If $\ell = 1$, then v is an eigenvector. Otherwise, $A^{\ell-1}(v) \neq 0$ is an eigenvector. □

(10) Define $T \in \mathcal{L}(\mathbb{C}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

Solution. You can solve also solve this by writing the matrix of T and proceeding from there.

Suppose $v = (x, y, z)$ is an eigenvector of T with eigenvalue λ. Then

$$T(v) = \lambda \cdot v \implies (x, y, z) = (\lambda \cdot 2y, 0, \lambda \cdot 5z).$$

So $x = \lambda \cdot 2y$, $y = 0$, and $z = \lambda \cdot 5z$. From this you can see that the only solution is $\lambda = 0$, and any vector with only x-coordinate is an eigenvector. Indeed, you can check that $(1, 0, 0)$ is an eigenvector with eigenvalue 0. ■

(11) Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

Proof. If $P(v) = \lambda \cdot v$, then

$$P^2(v) = P(P(v)) = P(\lambda \cdot v) = \lambda \cdot P(v) = \lambda \cdot \lambda \cdot v = \lambda^2 \cdot v.$$

On the other hand, $P^2(v) = P(v) = \lambda \cdot v$. So

$$\lambda v = \lambda^2 v \implies (\lambda - \lambda^2) \cdot v = 0 \implies \lambda - \lambda^2 = 0 \implies \lambda = \lambda^2.$$

You can also do this problem by a determinant argument. □