Problem Set 5

Matrix Theory & Linear Algebra II

(1) Show that if p is a polynomial with real coefficients and a complex root λ, then $\bar{\lambda}$ is also a root. Conclude that if λ is an eigenvalue of a matrix T with real coefficients, then so is $\bar{\lambda}$.

Solution. For the first part, consider a polynomial with *real* coefficients.

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

If w is a complex root of p, i.e.

$$a_0 + a_1 w + a_2 w^2 + \cdots + a_n w^n = 0$$

then conjugating this equation yields

$$a_0 + a_1 \bar{w} + a_2 \bar{w}^2 + \cdots + a_n \bar{w}^n = 0$$

because conjugation doesn't do anything to a real number. Hence if λ is a root of $det(T - cI)$ (a polynomial with real coefficients), then so is $\bar{\lambda}$. (There is also a minimal polynomial argument you could make.) ■

(2) Find the (real or complex) eigenvalues and eigenvectors of the following matrices. Diagonalize[1] each matrix if possible by finding a basis of eigenvectors.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

(3) (a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.
(b) Prove that T and T^{-1} have the same eigenvectors.

Solution. (a) If $Tv = \lambda v$ then

$$T^{-1}Tv = T^{-1}(\lambda v) \implies v = \lambda T^{-1} v \implies \frac{1}{\lambda} T^{-1} v.$$

So v is an eigenvector of T^{-1} as well, but with eigenvalue $\frac{1}{\lambda}$.

(b) Because of (a) it suffices to show that 0 is not an eigenvector. Indeed, if $Tv = 0 \cdot v$, then

$$Tv = 0 \implies T^{-1}Tv = T0 \implies v = 0.$$

■

[1] To diagonalize M is to find a diagonal matrix D and a matrix S such that $M = SDS^{-1}$

(4) Given the matrix:

$$A = \begin{bmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

(a) Find its eigenvalues. Is it possible to find the eigenvalues without computing?

(b) Is this matrix diagonalizable? Find out without computing anything.

(c) If the matrix is diagonalizable, diagonalize it.

Solution. (a)

■

(5) Show that if an operator $T : V \to V$ is not invertible, then 0 is an eigenvalue of T.

Solution. Any nonzero $v \in \text{null}(T)$ is an eigenvector with eigenvalue 0. ■

(6) Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$. Assume that, with respect to some basis (v_1, v_2, v_3, v_4) of V, we have:

$$\mathcal{M}(T) = \begin{pmatrix} 2+i & 3 & -1 & 4 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 2+i & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(a) Find the eigenvalues of T.

(b) Determine if T is invertible.

(c) Explain why, for each $k \in \{1, 2, 3, 4\}$, we have:

$$T(\text{span}(v_1, \ldots, v_k)) \subseteq \text{span}(v_1, \ldots, v_k).$$

Solution. (a) They are the four entries in the diagonal of T.

(b) If T was invertible, then 0 would be an eigenvalue (see question 5). But 0 is not an eigenvalue, so T is invertible.

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(7) (a) Give an example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.

(b) Suppose $\mathbb{F} = \mathbb{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

(a) The $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z, w) = (w, 0)$. Note that $T^2 = 0$, and the zero matrix is diagonal.

(b) This follows from (3b) and the fact that "diagonalizable" means "having a basis of eigenvectors".

(8) Define $T \in \mathcal{L}(P_4(\mathbb{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Solution. Let $p(x) = a + bx + cx^2 + dx^4$. Then

$$(Tp)(x) = x \cdot (b + 2cx + 4dx^3) = bx + 2cx^2 + 4dx^3$$

The eigenvalue equation $Tp = \lambda p$ becomes

$$bx + 2cx^2 + 4dx^3 = \lambda a + \lambda bx + \lambda cx^2 + \lambda dx^4.$$

If $\lambda = 0$, then a is a free variable. Hence $p(x) = a$ is an eigenvector with eigenvalue 0.

If $\lambda \neq 0$, then comparing the coefficient of x we conclude that $\lambda = 1$. Hence b is a free variable. We also conclude that $c = d = 0$. Indeed, $p(x) = b$ is an eigenvector with eigenvalue 1.

These are all eigenvectors. ■

(9) Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

Solution. Let α, β, and γ be the eigenvalues of u, v, and $u + v$, respectively. If $\alpha = \beta$, then $u + v$ is already an eigenvector:

$$T(u + v) = Tu + Tv = \alpha u + \alpha v = \alpha \cdot (u + v).$$

Otherwise $\alpha \neq \beta$, and in particular u and v are linearly independent. Then

$$T(u + v) = Tu + Tv \implies \gamma u + \gamma v = \alpha u + \beta v \implies (\alpha - \gamma)u + (\beta - \gamma)v = 0.$$

Hence $\alpha - \gamma = 0$ and $\beta - \gamma = 0$ by linearly independence, so $\alpha = \gamma = \beta$. Contradiction. ■

(10) Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity

Solution. Every vector in a basis $e_1, \ldots, e_n$ of V is linearly independent and an eigenvector, so they must correspond to the same eigenvalue λ. Hence for $v = a_1 e_1 + \cdots + a_n e_n$ we see that

$$Tv = a_1 Te_1 + \cdots + a_n Te_n = a_1 \lambda e_1 + \cdots + a_n \lambda e_n = \lambda(a_1 e_1 + \cdots + a_n e_n) = \lambda v.$$

So $T = \lambda \cdot I$. ■