Solutions to Problem Set 6

Matrix Theory & Linear Algebra II

Winter 2025

In this problem set, any vector space V comes equipped with an inner product.

(1) Choose one of the inner products defined in Example 6.3 of the book. Check that they are indeed inner products, i.e. show that the axioms are satisfied.

(2) Let $e_1, \dots, e_n$ be an orthonormal basis, and $v = a_1 e_1 + \cdots + a_n v_n$. Show that $a_i = \langle v, e_i \rangle$.

Solution. Since the basis is orthonormal,

$$\langle e_i, e_j \rangle = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} .$$

Then

$$\begin{aligned} \langle v, e_i \rangle &= \langle a_1 e_1 + \cdots + a_n v_n, e_i \rangle & \\ &= \langle a_1 e_1, e_i \rangle + \cdots + \langle a_n e_n, e_i \rangle & \text{linearity} \\ &= a_1 \langle e_1, e_i \rangle + \cdots + a_n \langle e_n, e_i \rangle & \text{linearity} \\ &= a_i \langle e_i, e_i \rangle & \text{all other terms are zero} \\ &= a_i & \langle e_i, e_i \rangle = 1 \end{aligned}$$

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(3) x Suppose that $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.

(4) Recall that in $\mathbb{R}^3$ it's true that

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \alpha,$$

where α is the angle between the vectors. There is no "angle" between vectors in a general vector space V, but in the presence of an inner product we can use this formula to define the angle between two vectors:

$$\cos(u, v) = \arccos\left(\frac{\langle u, v \rangle}{\|u\| \|v\|}\right).$$

(a) Show that this formula is well-defined. (What is the domain of arccos?)

(b) Using the inner product $\langle f, g \rangle = \int_0^1 f(x) g(x) dx$ on $C^0([0,1])$, calculate the angle between x, e^x, and $\sin x$.

Solution. (a) The domain of arccos is $[0, 1]$, and because of Cauchy-Shwarz:

$$\langle u, v \rangle \leq \|u\| \cdot \|v\| \implies \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|} \leq 1,$$

so the formula is well-defined.

(b) For instance:

$$\|e^x\| = \sqrt{\langle e^x, e^x} = \sqrt{\int_0^1 e^{2x} dx} = \sqrt{\frac{1}{2}(e^2 - 1)}$$
$$\|1\| = \sqrt{\langle 1, 1, \rangle} = \sqrt{\int_0^1 1 dx} = 1$$
$$\langle e^x, 1\rangle = \int_0^1 e^x \cdot 1 dx = e - 1.$$

So

$$\begin{aligned} \cos(e^x, 1) &= \arccos\left(\frac{\langle e^x, 1\rangle}{\|e^x\|\|1\|}\right) \\ &= \arccos\left(\frac{e-1}{\sqrt{(e^2-1)/2}}\right) \\ &\approx 0.28\,\text{rad} \approx 16^\circ. \end{aligned}$$

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(5) Let $\mathcal{B} = e_1, \ldots, e_n$ be a basis[1] for V, and define the matrix A whose entries are $a_{ij} = \langle e_i, e_j\rangle$. Show that, for $u, v \in V$,

$$\langle u, v\rangle = [u]_{\mathcal{B}}^{\intercal} \cdot A \cdot \overline{[v]}_{\mathcal{B}},$$

where $\overline{[v]}_{\mathcal{B}}$ denotes the vector with the conjugate of coordinates of v as its entries.

Solution. Recall that $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ corresponds to the expansion of v in the basis, i.e. $v = c_1 e_1 + \cdots + c_n e_n$. Similarly, let $[u]_{\mathcal{B}} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

We can calculate

$$\begin{aligned} \langle u, v\rangle &= \langle b_1 e_1 + \cdots + b_n e_n, c_1 e_1 + \cdots + c_n e_n\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} b_i \overline{c_j} \langle e_i, e_j\rangle && \text{applying linearity on both entries} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} b_i a_{ij} \overline{c_j} && \text{definition of } a_{ij} \\ &= [u]^{\intercal} A \overline{[v]}_{\mathcal{B}} && \text{matrix multiplication} \end{aligned}$$

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(6) In $\mathbb{R}^3$ with the usual dot product, find an orthonormal basis for

$$U = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}\right).$$

[1] The original statement was wrong - this question is trivial on an orthonormal basis.

Solution. Let $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $v_2 = \begin{bmatrix}2\\6\\0\end{bmatrix}$. Recall the orthornormalization procedure:

$$\begin{aligned} u_1 &= v_1, & e_1 &= \frac{u_1}{\|u_1\|} \\ u_2 &= v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle}, & e_2 &= \frac{u_2}{\|u_2\|} \end{aligned}$$

Note that $\langle v_1, v_1\rangle = \langle v_1, v_2\rangle = 14$, so $e_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}$. Then

$$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} = \begin{bmatrix}2\\6\\0\end{bmatrix} - \frac{14}{14}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\4\\-3\end{bmatrix}.$$

Normalizing:

$$e_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix}1/\sqrt{26}\\4/\sqrt{26}\\-3/\sqrt{26}\end{bmatrix}.$$

So an orthonormal basis is $\left\{\begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}, \begin{bmatrix}1/\sqrt{26}\\4/\sqrt{26}\\-3/\sqrt{26}\end{bmatrix}\right\}$. ■

(7) Consider the inner product space $C([0,2])$ with the inner product given by

$$\langle f, g\rangle = \int_0^2 f(x)g(x)dx.$$

Use Gram-Schmidt to find an orthogonal basis for $\operatorname{span}(1, x, x^2)$.[2]

Solution. Let $v_1 = 1$, $v_2 = x$, and $v_3 = x^2$. Recall the orthonormalization procedure:

$$\begin{aligned} u_1 &= v_1, & e_1 &= \frac{u_1}{\|u_1\|} \\ u_2 &= v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle}u_1, & e_2 &= \frac{u_2}{\|u_2\|} \\ u_3 &= v_3 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle}u_2, & e_3 &= \frac{u_3}{\|u_3\|} \end{aligned}$$

Then this question is like the previous one, done step by step, just make sure you calculate the inner products using the definition above. ■

(8) In $\mathbb{C}^3$ with the complex dot product, find an orthonormal basis for

$$U = \operatorname{span}\left(\begin{bmatrix}0\\i\\2\end{bmatrix}, \begin{bmatrix}1\\i+1\\3i+2\end{bmatrix}\right).$$

Solution. This will be like question 6, but you have to use the complex inner product. ■

[2] There was a typo in the original question.