# Problem Set 7

Matrix Theory & Linear Algebra II

Winter 2025

In this problem set, any vector space $V$ comes equipped with an inner product. The field $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$, unless specified. Ensure you do at least the first page of this document.

(1) True or false:

(a) Every unitary operator $U : X \to X$ is normal.

(b) A matrix is unitary if and only if it is invertible.

(c) If two matrices are unitarily equivalent, then they are also similar.

(d) The sum of self-adjoint operators is self-adjoint.

(e) The adjoint of a unitary operator is unitary.

(f) The adjoint of a normal operator is normal.

(g) If all eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.

(h) If all eigenvalues of a normal operator are 1, then the operator is identity.

(i) A linear operator may preserve norm but not the inner product.

(2) Four of the following matrices are diagonalizable. Which ones and why?

(a)
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 2 & i & 0 \\ -i & 3 & 4 \\ 0 & 4 & -1 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1+i & 2 & -i \\ 0 & 3-i & 4 \\ 0 & 0 & 5+2i \end{bmatrix}$$

(d)
$$\begin{bmatrix} 0.5 & -2 & 3 & 1 \\ -1 & 4.2 & 0 & 3.5 \\ 2 & -0.5 & 1.3 & 2.2 \\ 4 & -3 & 2 & -1 \end{bmatrix}$$

(e)
$$\begin{bmatrix} 2 & 1 & 0 & -1 & 3 \\ 1 & 4 & 2 & 0 & -2 \\ 0 & 2 & 5 & 1 & 4 \\ -1 & 0 & 1 & 3 & 2 \\ 3 & -2 & 4 & 2 & 6 \end{bmatrix}$$

(3) Check that the following real matrices are orthogonal and/or self-adjoint, and orthogonally diagonalize them. In other words, find orthonormal bases of eigenvectors in each case.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \quad C = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In each case give a geometric interpretation of the transformation.

(4) Prove the following properties of adjoint operators. You can do this from the definition, or by checking properties on a basis.

(a) $(S+T)* = S^* + T*$.
(b) $(\lambda \cdot T)^* = \overline{\lambda} \cdot T^*$.
(c) $(T^*)^* = T$
(d) $(ST)^* = T^* S^*$
(e) if $T$ is invertible, then $(T^{-1})^* = (T^*)^{-1}$
(f) if $\lambda$ is an eigenvalue of $T$, then $\overline{\lambda}$ is an eigenvalue of $T^*$.

(5) Show that $\ker T = (\operatorname{range} T^*)^{\perp}$.

(6) Let $T : V \to V$ be a self-adjoint operator. Show that if $\lambda_1 \neq \lambda_2$ are distinct eigenvalues, then the corresponding eigenvectors are orthogonal. Use this and the Spectral Theorem to conclude that a self-adjoint operator has an orthonormal basis of eigenvectors.

(7) An operator $T : V \to W$ is an *isometry* if $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$. Show that if $\dim V = \dim W$ then any isometry is invertible.

(8) An invertible operator $T : V \to V$ is *unitary* if it is an isometry (in particular, $T$ is invertible). Show that an operator is unitary if and only if $TT^* = T^*T = I$ (i.e. $U^{-1} = U^*$).

(9) Using the previous exercise, explain the following assertion: "Unitary and orthogonal operators are the operators that preserve angles and distances."

(10) Let $U$ be a $2 \times 2$ orthogonal matrix with $\det U = 1$. Prove that $U$ is a rotation matrix.

(11) Let $U$ be a $3 \times 3$ orthogonal matrix with $\det U = 1$. Prove that

(a) 1 is an eigenvalue of $U$.
(b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis, such that $U\mathbf{v}_1 = \mathbf{v}_1$ (remember, that 1 is an eigenvalue), then in this basis the matrix of $U$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

where $\alpha$ is some angle.

(12) Let $A$ be an $m \times n$ matrix. Show that

(a) $A^*A$ is self-adjoint.
(b) The eigenvalues of $A^*A$ are non-negative.
(c) $A^*A + I$ is invertible.

(13) Prove that a normal operator with whose eigenvalues satisfy $\|\lambda_k\| = 1$ is unitary. Hint: diagonalize.

(14) For $\mathbb{F} = \mathbb{R}$, show that self-adjoint operators form a subspace of $\mathcal{L}(V)$.(*Hint:* this was in your 2nd WebWork homework.) Show that this is false for $\mathbb{F} = \mathbb{C}$.