Problem Set 7

Matrix Theory & Linear Algebra II

Winter 2025

In this problem set, any vector space V comes equipped with an inner product. The field $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$, unless specified. Ensure you do at least the first page of this document.

(1) True or false:

(a) Every unitary operator $U : X \to X$ is normal.
(b) A matrix is unitary if and only if it is invertible.
(c) If two matrices are unitarily equivalent, then they are also similar.
(d) The sum of self-adjoint operators is self-adjoint.
(e) The adjoint of a unitary operator is unitary.
(f) The adjoint of a normal operator is normal.
(g) If all eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.
(h) If all eigenvalues of a normal operator are 1, then the operator is identity.
(i) A linear operator may preserve norm but not the inner product.

Solution. To each *false* assertion, there are *several* counterexamples. I'll give only one for each. Yours might be very different.

(a) T, for being normal means $NN^* = N^*N$, and unitary is the stronger condition $UU^* = U^*U = I$.
(b) F, for instance $2I$ is invertible (with inverse $\frac{1}{2}I$) but not unitary (its columns are not normalized).
(c) T, being unitarily equivalent means being similar via an unitary matrix.
(d) T, let T be an unitary operator, i.e. $TT^* = T^*T = I$, and $S = T^*$. Then S is unitary if $SS^* = S^*S = I$. But $S^* = (T^*)^* = T$. So the desired equations are $T^*T = TT^* = I$, which we already had for T.
(e) T, similar to the previous item.
(f) F, the only eigenvalue of the shear matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is 1 (for it's upper triangular), but its columns are neither orthogonal nor normalized, so this is not orthogonal/unitary.
(g) T, let N be a normal operator and hence you can find a basis of eigenvectors $e_1, \cdots, e_n$. In this basis N is diagonal, but in fact an identity matrix! So N is an identity transformation.

(h) F, as we saw in class the latter follows from the former. In the real case, this is a consequence from the identity

$$\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2.$$

In the complex case, it follows from the identity

$$\langle x, y\rangle = \|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2.$$

■

(2) Four of the following matrices are diagonalizable. Which ones and why?

(a)
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 2 & i & 0 \\ -i & 3 & 4 \\ 0 & 4 & -1 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1+i & 2 & -i \\ 0 & 3-i & 4 \\ 0 & 0 & 5+2i \end{bmatrix}$$

(d)
$$\begin{bmatrix} 0.5 & -2 & 3 & 1 \\ -1 & 4.2 & 0 & 3.5 \\ 2 & -0.5 & 1.3 & 2.2 \\ 4 & -3 & 2 & -1 \end{bmatrix}$$

(e)
$$\begin{bmatrix} 2 & 1 & 0 & -1 & 3 \\ 1 & 4 & 2 & 0 & -2 \\ 0 & 2 & 5 & 1 & 4 \\ -1 & 0 & 1 & 3 & 2 \\ 3 & -2 & 4 & 2 & 6 \end{bmatrix}$$

Solution. Note that

- (a) is diagonalizable because it is orthogonal. Alternatively, you can find out that the eigenvalues are ± 1, distinct, diagonalizable.
- (b) is diagonalizable because it is self-adjoint (equal to its conjugate transpose).
- (c) is diagonalizable because it has three distinct eigenvalues (sitting in the diagonal, since it is an upper triangular matrix).
- (e) is diagonalizable because it is symmetric.

■

(3) Check that the following real matrices are orthogonal and/or self-adjoint, and orthogonally diagonalize them. In other words, find orthonormal bases of eigenvectors in each case.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \quad C = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In each case give a geometric interpretation of the transformation.

Solution. The first two matrices are orthogonal, as their columns are orthogonal and normalized. The first and last matrices are also self-adjoint, for they are symmetric.

(I'll give a geometric solution for A, but you can also do it by solving the appropriate linear system.)

The first transformation exchanges the x and y axes, corresponding to a reflection across the line $y = x$. The vectors along the line $y = x$ are fixed by this transformation, while the vectors on the lines $y = -x$ sent to their inverses. Hence $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue -1. These vectors are orthogonal, but not normalized. To normalize them just divide by their lengths:

$$e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad e_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Thus we can write the change of basis matrix $S = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. This is an orthogonal matrix, so $S^{-1} = S^{\intercal}$ (as you can check). Hence $A = SDS^{\intercal}$, where $D = \begin{bmatrix} 1 & 0//0 & -1 \end{bmatrix}$.

You can check that the eigenvalues of B and C are distinct, hence the corresponding eigenvectors will be automatically orthogonal. So you can just find an eigenbasis however you want by simply normalizing them. In this basis you can write the change of basis matrix S, which will be orthogonal, hence its inverse will be $S^{-1} = S^{\intercal}$.

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(4) Prove the following properties of adjoint operators. You can do this from the definition, or by checking properties on a basis.

(a) $(S + T)* = S^* + T*$.
(b) $(\lambda \cdot T)^* = \overline{\lambda} \cdot T^*$.
(c) $(T^*)^* = T$
(d) $(ST)^* = T^* S^*$
(e) if T is invertible, then $(T^{-1})^* = (T^*)^{-1}$
(f) if λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^*.

Solution. (a)-(e) are in the book (Proposition 7.5, pp. 230)

For (f), if λ is an eigenvalue of T then $T - \lambda I$ is invertible. Let S denote the inverse, i.e.

$$(T - \lambda I)S = S(T - \lambda I) = I.$$

Using (b), (c) and (d), conjugating these equations yields

$$S^*(T^* - \overline{\lambda} I) = (T^* - \overline{\lambda} I)S^* = I,$$

so $T^* - \overline{\lambda} I$ is invertible, with inverse S^*. Hence $\overline{\lambda}$ is an of T^*. ■

(5) Show that $\ker T = (\operatorname{range} T^*)^{\perp}$.

Solution.

$$\begin{aligned} Tv = 0 &\iff 0 = \langle Tv, u \rangle && \text{for all } u \in V \\ &\iff 0 = \langle v, T^* u \rangle && \text{definition of } T^* \\ &\iff v \in (\operatorname{range} T^*)^{\perp} && \text{by definition} \end{aligned}$$

■

(6) Let $T : V \to V$ be a self-adjoint operator. Show that if $\lambda_1 \neq \lambda_2$ are distinct eigenvalues, then the corresponding eigenvectors are orthogonal. Use this and the Spectral Theorem to conclude that a self-adjoint operator has an orthonormal basis of eigenvectors.

Solution. We saw the proof of the first claim in class (see the book if needed). For the second, find a basis of eigenvectors using the spectral theorem, which is particular gives a basis on each eigenspace. Orthonormalize those sub-bases. The resulting list of vectors is already orthonormalized too because of the first part of this question. ■

(7) An operator $T : V \to W$ is an *isometry* if $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$. Show that if $\dim V = \dim W$ then any isometry is invertible.

Solution. It suffices to show that $\ker T = 0$. Indeed, suppose $Tv = 0$. Then, for all $w \in V$,

$$\langle v, w \rangle = \langle Tv, Tw \rangle = \langle 0, Tw \rangle = 0.$$

So $v = 0$, as we wanted. ■

(8) An invertible operator $T : V \to V$ is *unitary* if it is an isometry (in particular, T is invertible). Show that an operator is unitary if and only if $TT^* = T^*T = I$ (i.e. $U^{-1} = U^*$).

Proof. Fix $v \in V$. For all $u \in V$, we have

$$\langle u, v \rangle = \langle Tu, Tv \rangle = \langle u, T^*Tv \rangle.$$

So $T^*Tv = v$, i.e. $T^*T = I$. We know, by hypothesis, that T is invertible. So multiply both sides of the equation by T^{-1} on the right:

$$T^*TT^{-1} = IT^{-1} \implies T^* = T^{-1}.$$

So in particular $I = TT^{-1} = TT^*$. □

(9) Using the previous exercise, explain the following assertion: "Unitary and orthogonal operators are the operators that preserve angles and distances."

Solution. See the definition of cosine between vectors in Problem Set 6. ■

(10) Let U be a 2×2 orthogonal matrix with $\det U = 1$. Prove that U is a rotation matrix.

Solution. Hint: let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and write the corresponding equations to the conditions in the statement (there are 4 of them). ■

(11) Let U be a 3×3 orthogonal matrix with $\det U = 1$. Prove that

(a) 1 is an eigenvalue of U.

(b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis, such that $U\mathbf{v}_1 = \mathbf{v}_1$ (remember, that 1 is an eigenvalue), then in this basis the matrix of U is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

where α is some angle.

(12) Let A be an $m \times n$ matrix. Show that

(a) A^*A is self-adjoint.

(b) The eigenvalues of A^*A are non-negative.

(c) $A^*A + I$ is invertible.

Solution. (a) $(A^*A)^* = A^*(A^*)^* = A^*A$

(b) Suppose $A^*Av = \lambda v$, with $v \neq 0$. Then

$$\langle Av, Av\rangle = \langle v, A^*Av\rangle = \langle v, \lambda \cdot v\rangle = \lambda \cdot \langle v, v\rangle \implies \lambda = \frac{\langle Av, Av\rangle}{\langle v, v\rangle}.$$

Both denominator and numerator are non-negative, hence so is the fraction.

(c) If $A^*A + I$ was not invertible, then -1 would be an eigenvalue, which it is not by (b).

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(13) Prove that a normal operator with whose eigenvalues satisfy $\|\lambda_k\| = 1$ is unitary. Hint: diagonalize.

Proof. In a basis of eigenvectors, the matrix D of such an operator is diagonal with λ_k's in the diagonal, and the matrix D^* of its adjoint is diagonal with $\overline{\lambda_k}$'s in the diagonal. Hence D^*D is diagonal with $\overline{\lambda_k}\lambda_k = |\lambda_k| = 1$ in the diagonal, i.e. $D^*D = I$. □

(14) For $\mathbb{F} = \mathbb{R}$, show that self-adjoint operators form a subspace of $\mathcal{L}(V)$.(*Hint:* this was in your 2nd WebWork homework.) Show that this is false for $\mathbb{F} = \mathbb{C}$.

Solution. Real self-adjoint operators correspond to symmetric matrices, which form a subspace.

Hermitian matrices are not a subspace. For instance the Pauli matrix $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is Hermitian, but multiplying this matrix by i results in the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which is not Hermitian.

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