Midterm (length: 2 hours)

Matrix Theory & Linear Algebra I

Summer 2025

Assume that you have to explain your reasoning even if the question doesn't explicitly asks you to.
Every scalar or variable in this exam is over the field of real numbers.

1. (30 points) In this question you are presented with three statements. *Each one of them is wrong.* For each one, give a short (one or two sentences) explanation of why.
   (a) Every system of linear equations with more equations than variables has a solution.
   (b) There exists a system of linear equations with exactly three solutions.
   (c) The equation $(v \times (v \times w)) \cdot v = 0$ is true for any two vectors v and w in $\mathbb{R}^3$.

2. (20 points) Consider the following system of linear equations.

$$\begin{cases} x - 4y - z + w = 3, \\ 2x - 8y + z - 4w = 9, \\ -x + 4y - 2z + 5w = -6. \end{cases}$$

   (a) Write the corresponding augmented matrix for the system.
   (b) Put the system in reduced row echelon form.
   (c) Solve the system in any way you want.

3. (30 points) Given a list of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$, consider the vector equation

$$x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + x_3\overrightarrow{v_3} = 0,$$

   where x_1, x_2, x_3 are scalars. The vectors are called *linearly independent* if the only solution to this equation is $x_1 = x_2 = x_3 = 0$.

   (a) Show that the vectors $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ and $\overrightarrow{v_3} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ are linearly independent.
   (b) Show that any three 2-dimensional vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ are **not** linearly independent. *Hint:* contrast the number of equations with the number of variables.

4. (20 points) Relative to a fixed origin O, the respective position vectors of three points A, B and C are

$$A = (4, 5, 5), \quad B = (4, 1, 2) \text{ and } C = (-1, 1, 2).$$

   (a) Determine, in component form, the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.
   (b) Determine the angles of the triangle ABC. What is its area?