A model structure for 2-fibrations

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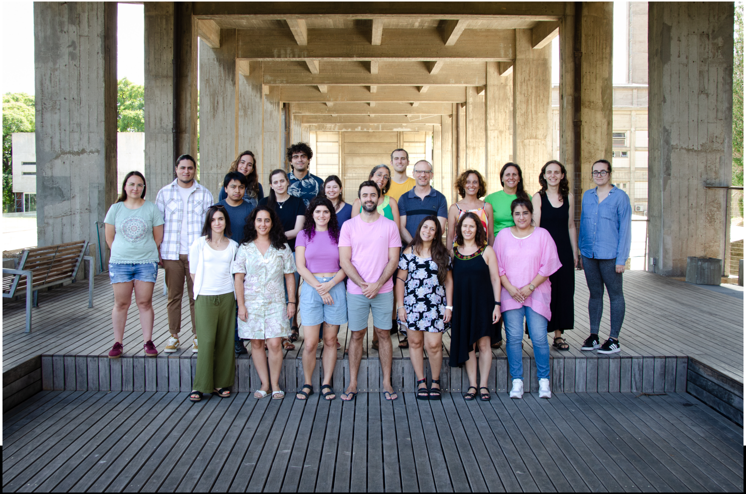

Definition. (Bacović, Buckley, Hermida)

A 2-functor $P : \mathcal{E} \to \mathcal{B}$ is a **2-fibration** if

- Every morphism $f : x \to Py$ has a cartesian lift.
- Every 2-cell $\alpha : f \Rightarrow Pg$ has a cartesian lift.
- the horizontal composite of any two cartesian 2-cells is cartesian.

Theorem. (Buckley)

There exists a triequivalence of tricategories

$$\mathsf{Un} : [\mathcal{B}, \mathbf{2Cat}] \cong \mathbf{2Fib}/\mathcal{B} : \int_{\mathcal{B}} .$$

Definition.

A model category $(\mathcal{M}, \mathrm{Cof}, \mathcal{W})$ is **cofibrantly generated** if there exists sets

$$\mathcal{I} \subseteq \mathrm{Cof} \quad \text{and} \quad \mathcal{J} \subseteq \mathrm{Cof} \cap \mathcal{W}$$

that generate Cof and $\mathrm{Cof} \cap \mathcal{W}$ via compositions, pushouts, and coproducts.

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that generate Cof and $\mathrm{Cof} \cap \mathcal{W}$ via compositions, pushouts, and coproducts. *Notation:* $\mathrm{Cof} = \mathrm{cell}(\mathcal{I})$, $\mathrm{Cof} \cap \mathcal{W} = \mathrm{cell}\mathcal{J}$.

Theorem. (Lack)

There exists a cofibrantly generated model structure on **2Cat** which models the homotopy theory of pseudofunctors.

Definition.

The **projective model structure** on $[\mathcal{B}, \mathbf{2Cat}]_{\mathsf{proj}}$ is the one whose fibrations and weak equivalences are defined objectivewise.

- Cofibrantly generated categories $(\mathcal{M}, \mathrm{cell}(\mathcal{I}), \mathcal{W})$ are nice because we can specify the fibrants by just a few lifting properties.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{I}\ni"'] & c \arrow[d] \\ \bullet \arrow[r] \arrow[ur, dashed, "\exists !"] & * \end{tikzcd} \iff c \text{ is fibrant}$$

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- However we have no control over the weak equivalences.

Suppose your fibrant objects are characterized via some lifting property, and that you know what you want the weak equivalences between them to be.

Theorem. (Guetta, Moser, Sarazola, Verdugo)

Sometimes you can equip $\mathcal{C}$ with a model structure

- with precisely those fibrant objects
- with those weak equivalences between those fibrant objects.

Definition.

A **marked 2-category** is a 2-category $\mathcal{B}$ with distinguished sets

- C_1 of *marked 1-cells* which contains all equivalences.
- C_2 of *marked 2-cells* which contains all invertible 2-cells.

Morphisms $F : (\mathcal{B}, C_1, C_2) \to (\mathcal{B}', C_1', C_2')$ are 2-functors preserving the markings, and they form a category $\mathbf{2Cat}^{\mathrm{marked}}$.

Proposition. [BNOSTV]

There exists a model structure on $\mathbf{2Cat}^{\text{marked}}/\mathcal{B}^\sharp$ such that

- the fibrant objects are 2-fibrations $P : (\mathcal{E}, C_1, C_2) \to \mathcal{B}^\sharp$ whose markings are the 1- and 2-cartesian cells.
- a morphism between fibrant objects
$$\begin{tikzcd} (P, C_1, C_2) \arrow[rr, "F"] \arrow[dr] & & (P', C_1', C_2') \arrow[dl] \\ & \mathcal{B}^\sharp & \end{tikzcd}$$
is a weak equivalence iff F it is a biequivalence.

Proposition. [BNOSTV]

There exists a Quillen equivalence

$$\Upsilon^+_{\mathcal{B}} : \mathbf{2Cat}/\mathcal{B}^\sharp \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} [\mathcal{B}^{\mathsf{coop}}, \mathbf{2Cat}]_{\mathrm{proj}} : \textstyle\int^+_{\mathcal{B}} \ .$$

There is also a notion of fibration of $(\infty, 2)$-categories.

Proposition. [BNOSTV]

There exists a diagram of right Quillen functors (up to levelwise w.e.)

$$\begin{tikzcd}
{[\mathcal{B}, \mathbf{2Cat}]_{\mathrm{proj}}} \arrow[rr, "\int_{\mathcal{B}}^{+}"] \arrow[d, "N_2^{+}"'] & & \mathbf{2Cat}^{\mathsf{marked}}/\mathcal{B}^{\sharp} \arrow[d, "N_2^{+}"] \\
{[\mathcal{B}, \mathbf{sSet}^{\mathsf{ms}}]} \arrow[rr] & & \mathbf{sSet}^{\mathsf{mb}}/N_2^{\mathsf{sc}}(\mathcal{B})^{\sharp}
\end{tikzcd}.$$

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