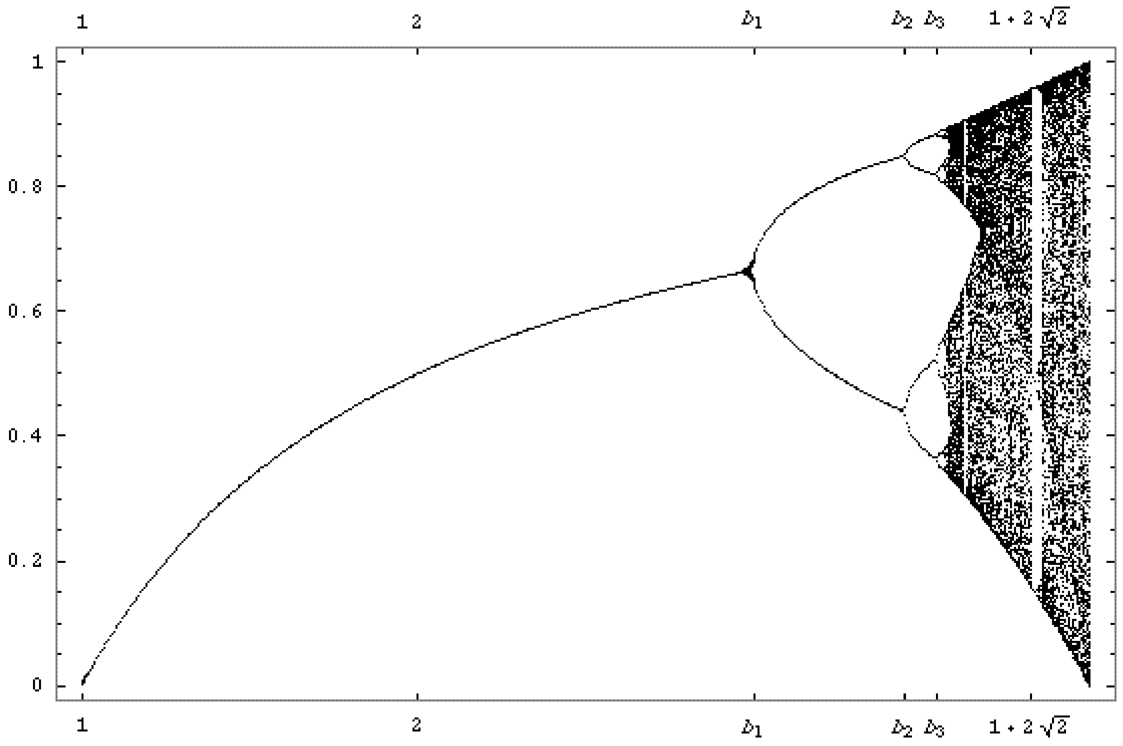
DANIEL TEIXEIRA

DAL HONOURS SEMINAR FALL 24’

CATASTROPHE!

SOME MOTIVATION: BIFURCATION THEORY

- https://youtu.be/D7m0pHEUfbw?list=PL8erL0pXF3JZqdlYIfTTyibOqSqwzRdVV&t=170



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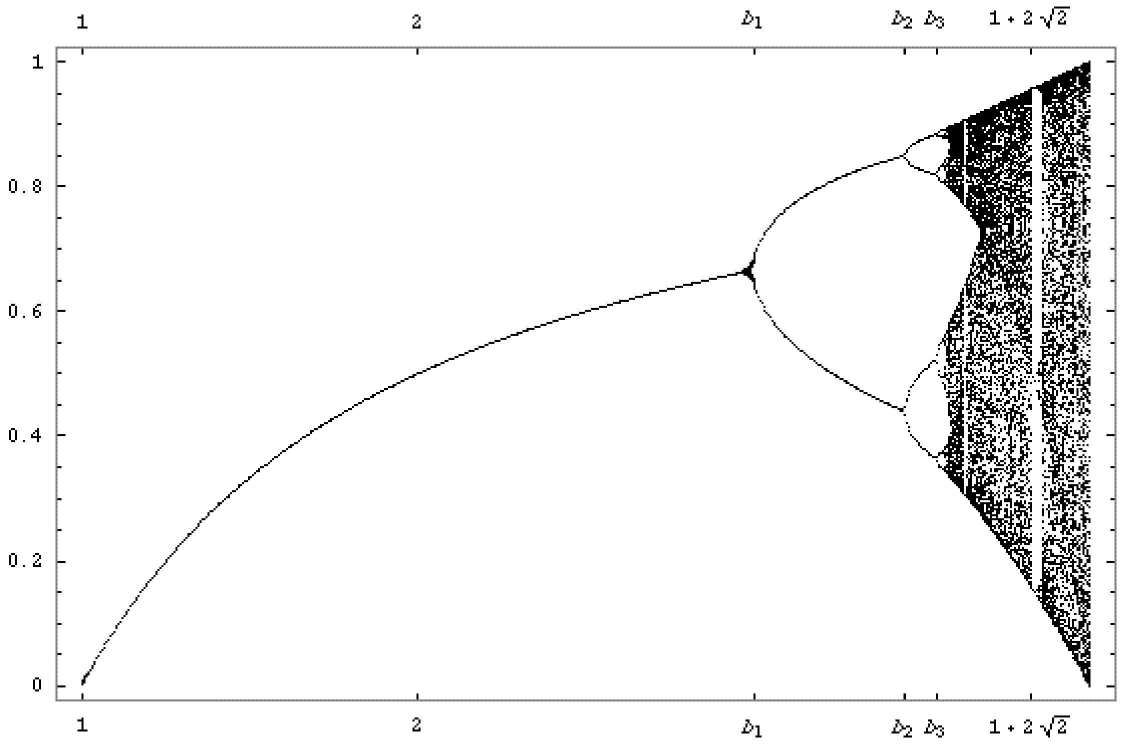


Goal: classify functions

$f: R^n \rightarrow R$

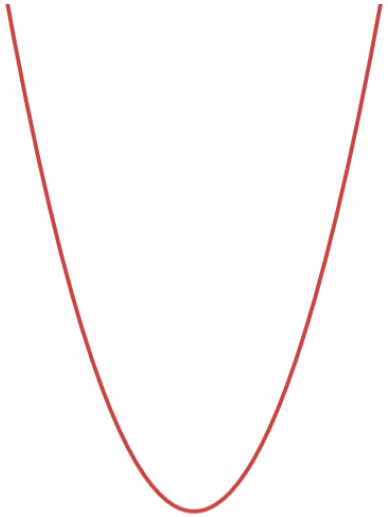
up to small perturbations.

This turns out to be extremely hands on in low dimensions.

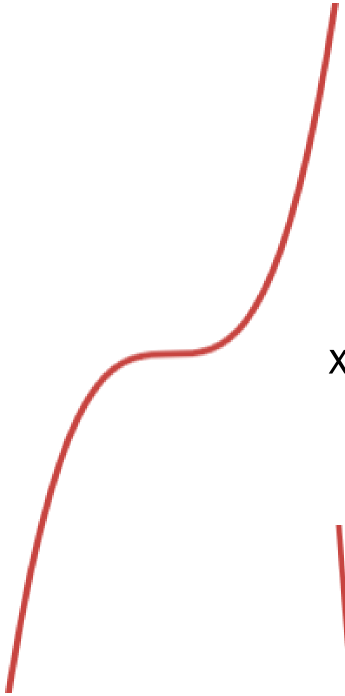
Surprising connection to higher category theory (very algebraic field!).

CRITICAL POINTS

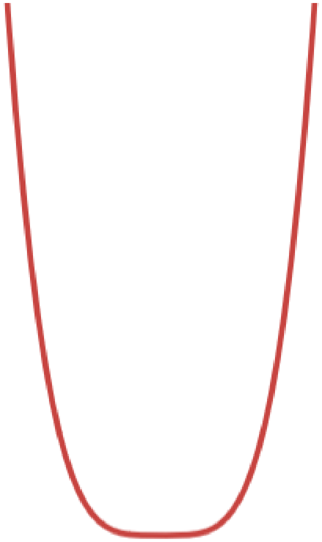
- A function f:R -> R has a critical point if f’ = 0.
- A critical point is *degenerate* if f’’(x) = 0.
- Otherwise, it is *non-degenerate.*



x^2 and cos(x) are non-degenerate at 0



x^3 and x^4 are degenerate at 0



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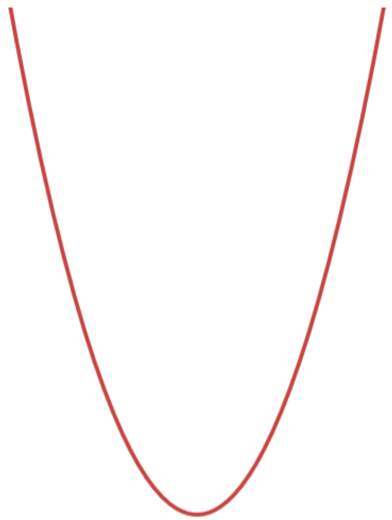
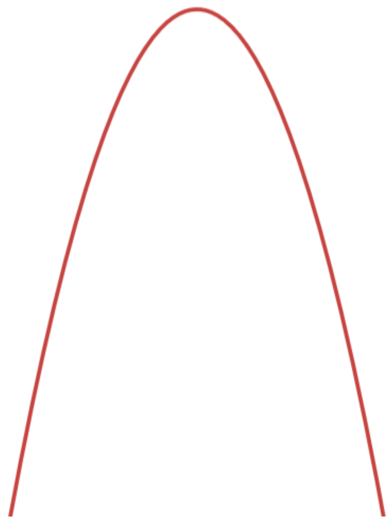


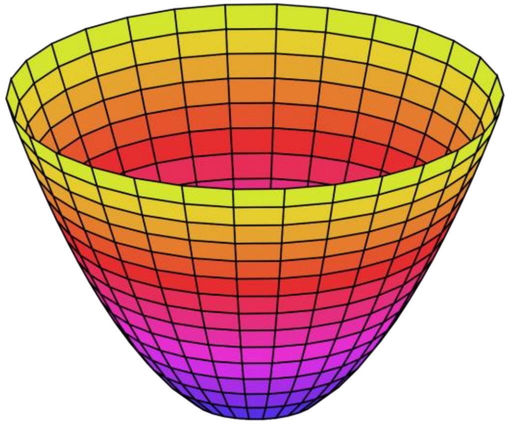
Theorem: up to a coordinate transformation, $\pm x^2$ are the only possible non-degenerate singularities.

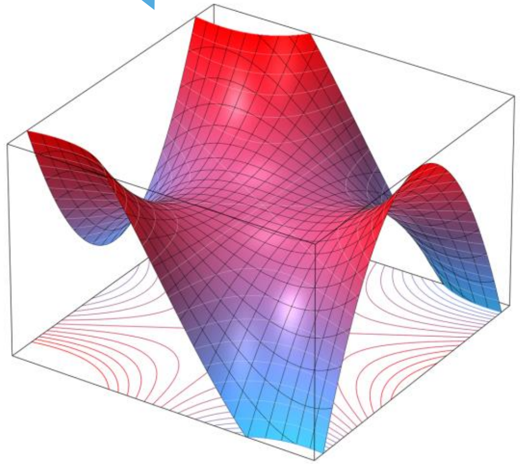


CRITICAL POINTS

- A function $f:R^n \to R$ has a critical point if df= 0.
- A critical point is *degenerate* if H(f) is non-invertible.
- Otherwise, it is *non-degenerate.*

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$





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Morse lemma: up to a coordinate transformation,

$$\pm x_1^2 \pm x_2^2 \pm \ldots \pm x_n^2$$

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are the only possible non-degenerate singularities.

What about degenerate critical points?

CRITICAL POINTS

- **Splitting lemma:** up to a coordinate transformation, any function $f:R^n \to R$ can be written as

$$f = \pm x_1^2 \pm x_2^2 \pm \ldots \pm x_m^2 + g(x_{m+1}, \ldots, x_n)$$

where $H(g) = 0$.

CRITICAL POINTS

- **Splitting lemma:** up to a coordinate transformation, any function $f:R^n \rightarrow R$ can be written as

$$f = \underbrace{\pm x_1^2 \pm x_2^2 \pm \ldots \pm x_m^2}_{\text{non-degenerate part}} + \underbrace{g(x_{m+1}, \ldots, x_n)}_{\text{degenerate part}}$$

where $H(g) = 0$.

CLASSIFICATION IN LOW DIMENSIONS

germ	codimension	corank	name
x^3	1	1	fold
$\pm x^4$	2	1	cusp
x^5	3	1	swallowtail
$\pm x^6$	4	1	butterfly
x^3+y^3	3	2	hyperbolic umbilic
x^3-xy^2	3	2	elliptic umbilic
X^2y+y^4	4	2	parabolic umbilic

MEANING OF THE CLASSIFICATION

- Classification *up to coordinate transformation*. i.e. classifying *germs* of smooth functions
- *Determinacy*: germs are classified by their Taylor expansions
- This is guaranteed my looking at germs with finite *codimension*.

MEANING OF THE CLASSIFICATION

- What happens to a germ g under perturbation?

Theorem: if G has codimension k, then for any perturbation G(x,t) there exist functions $f_1, \ldots, f_k$ such that $G = g + u_1 f_1 + \ldots + u_k f_k$.

MEANING OF THE CLASSIFICATION

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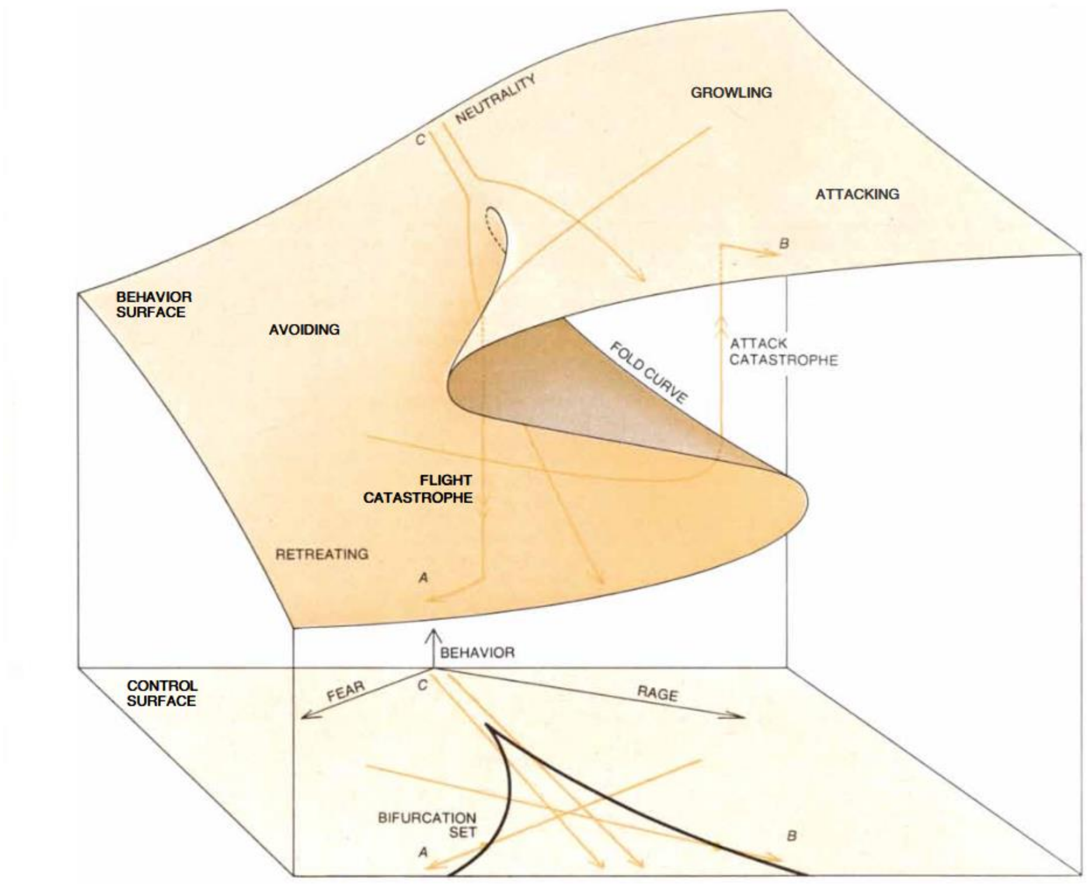
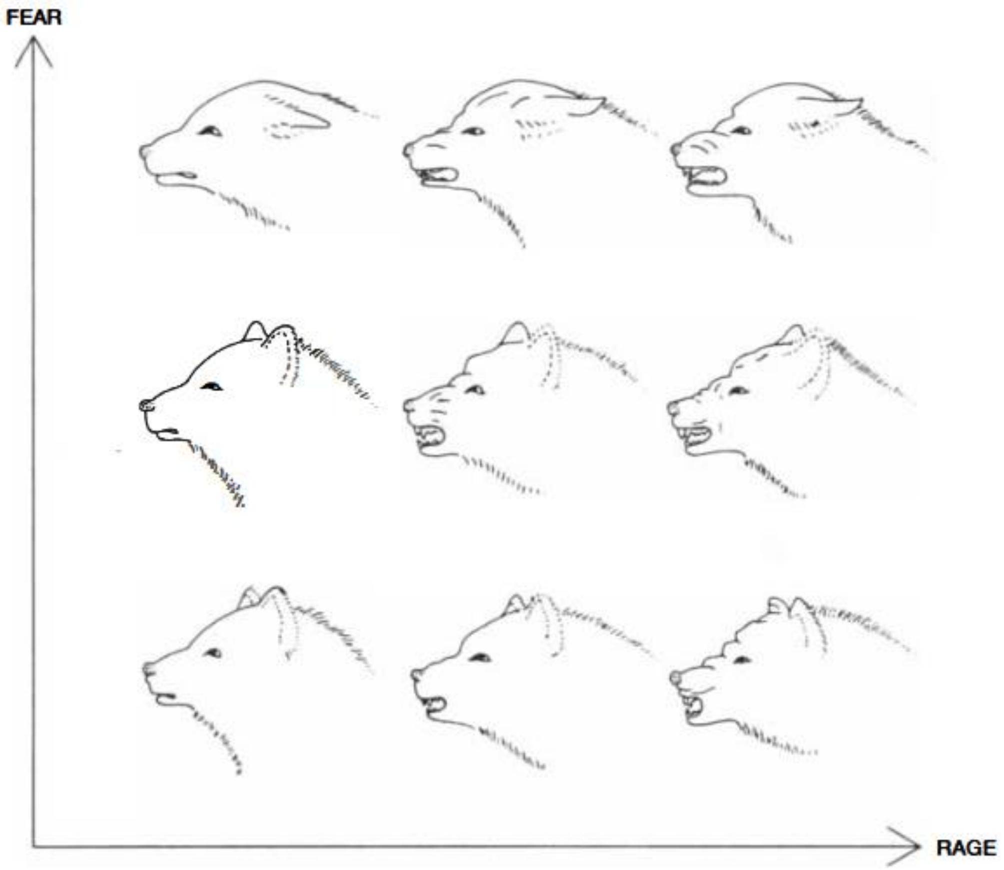
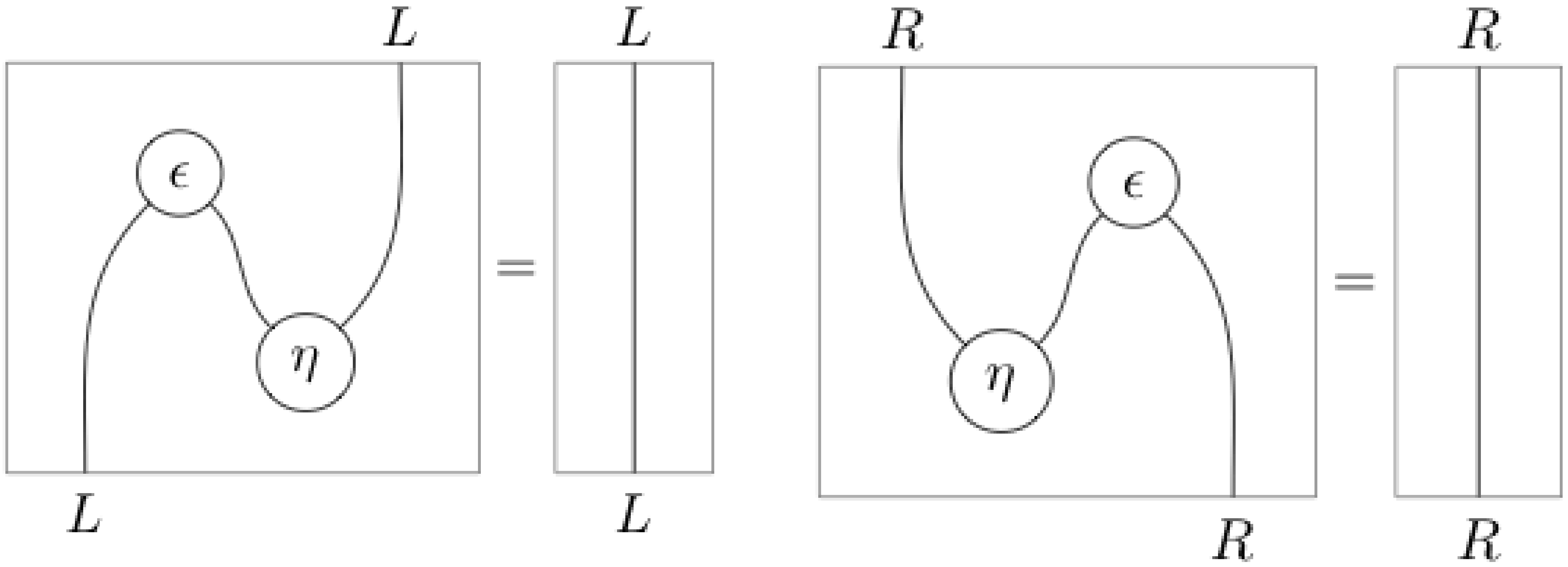
germ	codimension	corank	name	unfolding
x^3	1	1	fold	$x^3 + ux$
$\pm x^4$	2	1	cusp	$x^4 + ux^2 + vx$
x^5	3	1	swallowtail	$x^5 + ux^3 + vx^2 + wx$
$\pm x^6$	4	1	butterfly	$x^6 + ux^4 + vx^3 + wx^2 + tx$
x^3+y^3	3	2	hyperbolic umbilic	$x^3+y^3+uxy+vx+wy$
x^3-xy^2	3	2	elliptic umbilic	$x^3-xy^2+u(x^2+y^2)+vx+wy$
x^2y+y^4	4	2	parabolic umbilic	$x^2y+y^4+ux^2+vy^2+wx+ty$

APPLICATIONS??





DUALITY IN HIGHER CATEGORIES



DUALITY IN HIGHER CATEGORIES

germ	codimension	corank	name	categorical dimension
x^3	1	1	fold	2
$\pm x^4$	2	1	cusp	3
x^5	3	1	swallowtail	4
$\pm x^6$	4	1	butterfly	5
x^3+y^3	3	2	hyperbolic umbilic	???
x^3-xy^2	3	2	elliptic umbilic	???
X^2y+y^4	4	2	parabolic umbilic	???

DUALITY IN HIGHER CATEGORIES

- **Conjecture:** for each germ of degenerate singularities in codimension n, corresponds a coherence diagram in (n+1)-categories with duals.

- **Hope:** the catastrophes give insights to the meaning of those coherence diagrams.

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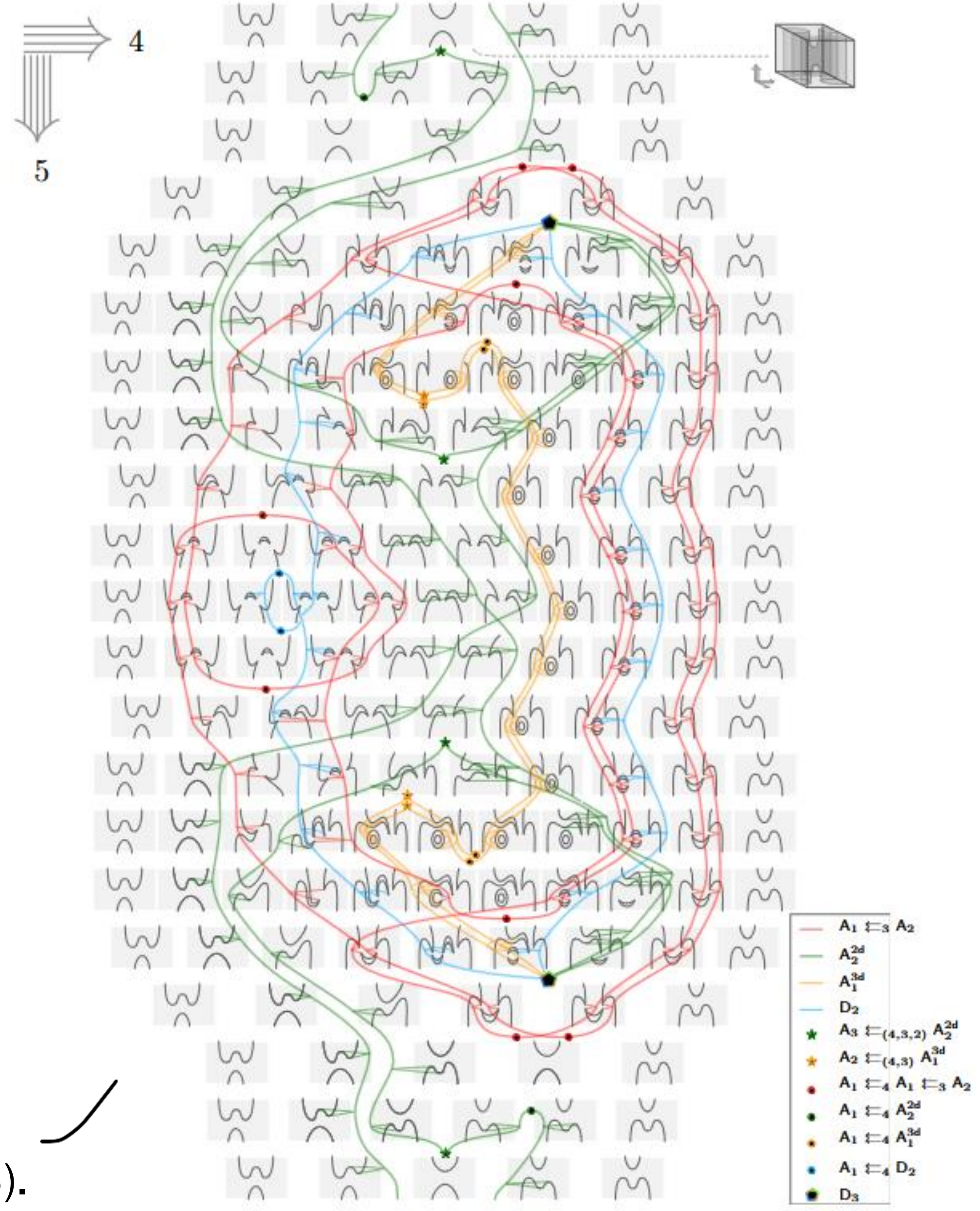
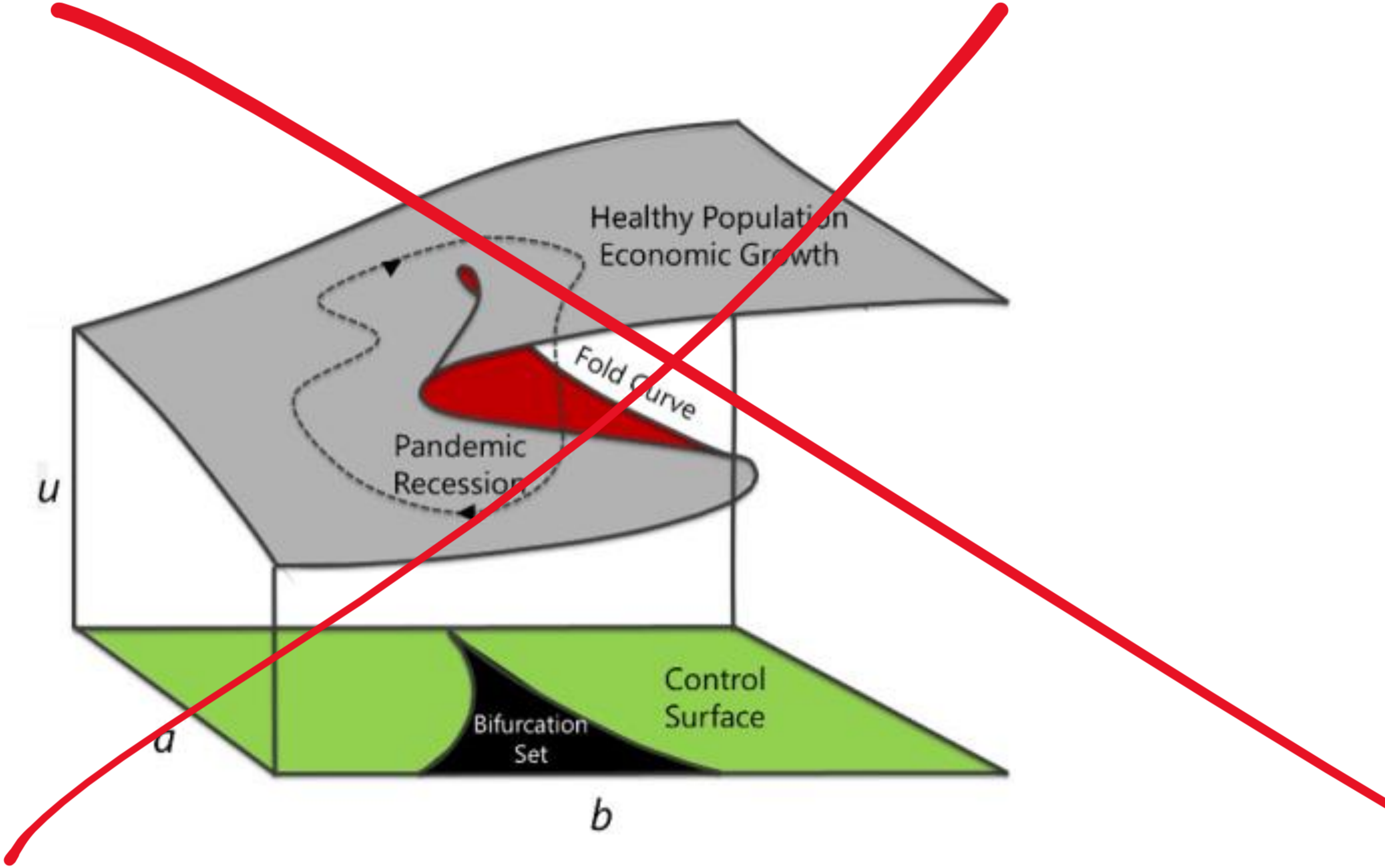


Manifold diagrams and tame tangles, C. Dorn & C. Douglas (2023).

THANK YOU!!


Healthy Population
Economic Growth
Fold Curve
Pandemic
Recession
u
Control
Surface
Bifurcation
Set
a
b