2-fibrations

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$P: P \to C$ fibration $\exists f: x \to y$ has a Cartesian lift $\in P$

Thm $\mathrm{Fib}(C) \underset{\int^{+}}{\overset{\text{take fibers}}{\rightleftarrows}} [C^{op}, \mathcal{C}at]$ is equiv

$f: x \to y$, $\exists!\, h$, $x \xrightarrow{f} y$, $h \downarrow$, $P_x \xrightarrow{Pf} P_y$, $\in P$

$\in P$, $\downarrow F$, C

Q. Can you encode fibrations w/ lifts!

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & P \arrow[d] \\ \bullet \arrow[r] & C \end{tikzcd} \subseteq \mathcal{C}at^{+}/C = \left(\begin{tikzcd} (C, E) \arrow[d, "F"] \\ (D, F) \end{tikzcd}\right)$$

$f: x \to Py$; $\mathcal{C}at^{+}$ = marked category; (C, E) ES/MC, (D, F) FS/MD

Goal: C = all marked, P = marked = cartesian

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & P \arrow[d] \\ \bullet \arrow[r] & C \end{tikzcd} \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & P \arrow[d] \\ \bullet \arrow[r] & C \end{tikzcd} = \text{cartesian } +$$

composition of cart is cart

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & X \arrow[d] \\ \bullet \arrow[r] & Y \end{tikzcd} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ \emptyset \arrow[r] & Y \end{tikzcd}$$

Thm (Lyne + Moriu) $\mathcal{C}at^{+}/C^{\#}$ (all marked), fibrants = fibrations

$$\begin{tikzcd} \mathcal{C}at^{+}/C^{\#} \arrow[d, "\int^{+}"] \arrow[r, "\text{localize}"] & \mathcal{C}at^{+}/C^{\#} \arrow[d, "\int^{+}"] \\ [C^{op}, \mathcal{C}at] \arrow[r, "\text{localize 1-morphisms}"] & [C^{op}, \mathcal{C}at] \end{tikzcd} \quad (\int F, \int \mathrm{Id}) \quad F: C^{op} \to \mathcal{C}at \mapsto \int^{C^{op}} \mathrm{Set}$$

$\int F \to C^{\#}$, $\int \mathrm{Id} \to C^{\#}$, $\int^{C^{op}} \mathrm{Set}$

Aside: $\mathcal{G}pd^{2}$: $\mathcal{G} = \mathrm{gpd}$, Set, $\pi_0 \mathcal{G} = h_0 \mathcal{G}$, C is a gpd $\leftrightarrow C \in \mathcal{G}pd$, $X = \mathrm{set}$; $\mathcal{G}pd \subseteq \mathcal{C}at$

$\mathcal{G}pd^{2}$: $h_2 \mathcal{G}$, $h_1 \mathcal{G}$, $h_0 \mathcal{G}$; $\mathcal{G}pd^{2} \subseteq \mathcal{G}pd$, $\mathcal{G}pd \subseteq \mathcal{S}$

(n, m): $n - m$ = height, m = categorically $(\mathrm{Set}, \mathrm{Set})$, $(1, 1)$, $\tau \in \mathcal{G}$, $(0, 1)$

2-fibrations

Def) $P \xrightarrow{F} B \in 2\mathcal{C}at$ is a 2-fibration if
- cartesian lifts of 1-morphisms
- locally a fibration $B(x, y) \to P_x$
- cartesian 2-morphisms compose horizontally

Def) $f: x \to y \in P$ is F-cartesian if

(I) $\hat{h}: \hat{x} \to y$, $\hat{g}$, $(\hat{h}, \hat{g})$ (II) h, $\hat{g}$, $\hat{g}'$, $(\hat{h}, \hat{\alpha})$, $(\hat{h}', \hat{\alpha}')$

$\alpha: Fg \circ h \Rightarrow Fg$, $\alpha': Fg' \circ h' \Rightarrow Fg'$

Thm (Beckley, 2014) (Hermida, 99) $2F: b(B) \underset{\int_B}{\overset{\text{take fibers}}{\rightleftarrows}} [B^{coop}, 2\mathcal{C}at]$ tri equivalence

(image uses localization)

(1) MS

$$\begin{tikzcd} 2\mathcal{C}at^{++}/B \arrow[r, shift left] \arrow[d] & [B^{coop}, 2\mathcal{C}at]_{\text{projective}} \arrow[l, shift left] \arrow[d] \\ (\rho, \rho)\mathcal{C}at^{++} \arrow[r, shift left] & [NB^{coop}, (\infty, 1)\mathcal{C}at] \arrow[l, shift left] \end{tikzcd}$$

(3) QE

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & P \arrow[d] \\ \bullet \arrow[r] & B \end{tikzcd} \quad \begin{tikzcd} (\bullet) \arrow[r] \arrow[d] & E \arrow[d] \\ (\bullet) \arrow[r] & B \end{tikzcd} \to \text{cartesian}$$

WAdj (no marks) $\to P$, $\downarrow$; WAdj (fully marked) $\to B$

C $\mathcal{C}at$: $\Sigma C \to \bullet \underset{\in 2\mathcal{C}at}{\overset{C}{\rightrightarrows}} \bullet$

$\Sigma C \to B$

$C \to B(x, y)$

$$\begin{tikzcd} \Sigma A \arrow[r] \arrow[d, "\Sigma F"'] & P \arrow[d, "F"] \\ \Sigma B \arrow[r] & B \end{tikzcd} \leftrightarrow \begin{tikzcd} A \arrow[r] \arrow[d, "F"'] & P(x, y) \arrow[d, "F"] \\ B \arrow[r] & B(Fx, Fy) \end{tikzcd} \quad F: P \to B$$

(1) Cartesian
(2) $\Sigma \to$ 2-cats
(3) horizontal marked, compose

$\to 2\mathcal{C}at^{++}/B$ ✓

$$2\mathcal{C}at^{++}/B^{\#} \underset{\leftarrow}{\overset{\to}{\rightleftarrows}} [B^{coop}, 2\mathcal{C}at]$$

shell

$(B, E_1, E_2) \to B[E_2^{-1}]$?

$\hookrightarrow B[E_1^{-1}]$! Dorette

$\mathrm{Bord}_{(d+1, d+2)}$, Bord, F, C

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\pi_n"] \arrow[d] & \bullet \arrow[d] \\ h_1 \mathcal{G} \arrow[r] & 2\mathcal{G}pd \end{tikzcd} \neq \begin{tikzcd} \mathcal{G} \arrow[r, "\pi_n"] \arrow[d] & \bullet \arrow[d] \\ K(\pi, 1) \arrow[r] & K(\pi_2, 3) \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[d, "\pi_0(C)"] \\ h_0(C) \end{tikzcd} \quad \begin{tikzcd} C \arrow[d] \\ \hat{C} \arrow[d] \end{tikzcd} \quad f: x \to y \quad [x] \xrightarrow{?} [y]$$