Categorical spectra, cohomology theories, and k-invariants

Gong show at: Summer School on Higher Structures @ Universität Hamburg

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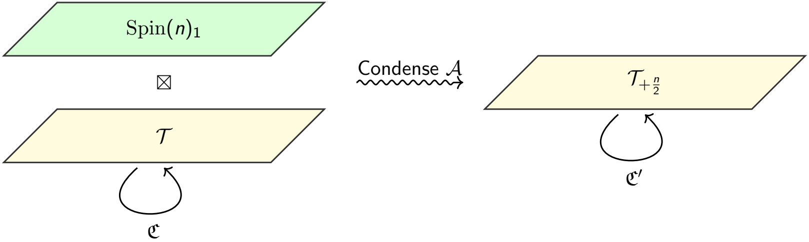
In [TY25] we study the following operation:



Symmetries of $\mathcal{T} \in$ a certain class of fusion 2-category $\mathfrak{C}$

$\rightsquigarrow$ classified by a cohomology theory
built categorically (supercohomology) [GW14]

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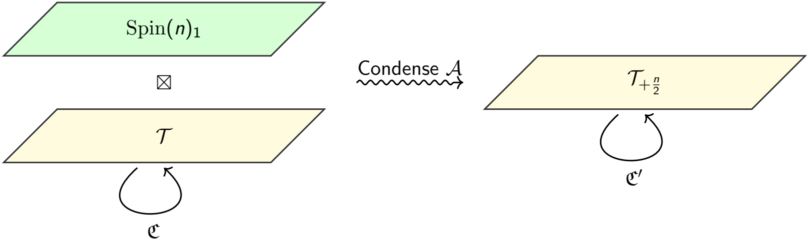

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Today: general story behind such cohomology theories.

Definition

A categorical spectrum is a sequence of (∞, ω)-categories $(*)$

$$\mathfrak{C}^0, \mathfrak{C}^1, \mathfrak{C}^2, \ldots$$

with equivalences $\mathfrak{C}^n \cong \Omega \mathfrak{C}^{n+1}$. [Ker24, Mas, Ste]

$(*)$ See [LH23, Lou24, SP] & Martina's talk next week.

Example

Delooping a symmetric monoidal category $\mathcal{C}^{\otimes}$ gives a categorical spectrum

$$\mathcal{C}, B\mathcal{C}, B^2\mathcal{C}, \ldots$$

A spectrum is called *connective* if it arises like this.

Underlying a categorical spectrum $\mathfrak{C}$ is an ordinary Ω-spectrum $\mathfrak{C}^\times$

Program

1. How can we understand cohomology theories arising from categorical spectra?
2. Which Ω-spectra underlie "intesting" categorical spectra?

In [Reu21] some results related to (2) & Kitaev's conjecture by Johnson-Freyd-Reutter are announced.

Example: reduced supercohomology

Reduced cohomology is a spectrum with $\pi_{-1} = \mathbb{Z}/2$ and $\pi_0 = \mathbb{C}^\times$ which appears in physics [GW14, GJF19].
The spectrum can be obtained by considering super vector spaces,

$$\mathbf{sVect}, B\mathbf{sVect}, B^2\mathbf{sVect}, \ldots$$

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and taking Picard groupoids:

$$r\mathrm{SH}^\bullet = (\mathbb{C}^\times, \mathbf{sVect}^\times, B\mathbf{sVect}^\times, \ldots)$$

Proposition

The cocycles in $r\mathrm{SH}^\bullet$ are given by

$$r\mathrm{SH}^n(X, \mathfrak{C}) \cong \begin{cases} \alpha \in Z^{n-1}(X, \pi_0) \\ \beta \in C^n(X, \pi_1) \text{ s.t. } d\beta = (-1)^{\mathrm{Sq}^2}\alpha \end{cases}.$$

The operation $(-1)^{\mathrm{Sq}^2} : H\mathbb{Z}/2 \to \Sigma^2 H\mathbb{C}^\times$ appears here because it is the k-invariant of $\mathbf{sVect}^\times$.

$$\begin{array}{ccc} \Sigma H\mathbb{C}^\times & \hookrightarrow & \mathbf{sVect}^\times \\ & & \downarrow \\ & & H\mathbb{Z}/2 \xrightarrow{(-1)^{\mathrm{Sq}^2}} \Sigma^2 H\mathbb{C}^\times \end{array}$$

Example: supercohomology

Supercohomology is a shift of the Picard groupoid **2sVect** to concentrate its homotopy groups in negative degrees

$$\mathrm{SH}^\bullet = (\mathbb{C}^\times, \textbf{sVect}^\times, \textbf{2sVect}^\times, B\textbf{2sVect}^\times, \dots)$$

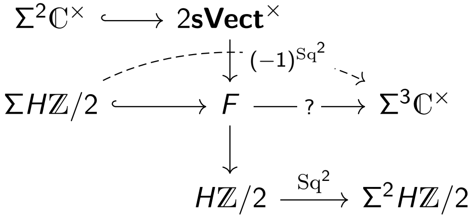
[JFR23] (also see David's course this week)

Fermionic strongly fusion 2-categories are classified by a group G_b, a class $\kappa \in \mathrm{H}^2(BG_b, \mathbb{Z}/2)$, and a cocycle in $\mathrm{SH}^{4+\kappa}(BG_b)$, a version of supercohomology twisted by κ.

Proposition

The cocycles in $\mathrm{SH}^\bullet$ are given by

$$\mathrm{SH}^n(X,\mathfrak{C}) \cong \begin{cases} \alpha \in Z^n(X,\pi_0) \\ \beta \in C^{n+1}(X,\pi_1) \text{ s.t. } d\beta = \mathrm{Sq}^2\alpha \\ \gamma \in C^{n+2}(X,\pi_2) \text{ s.t. } d\gamma = (-1)^{\mathrm{Sq}^2\beta} + f(\alpha) \end{cases} .$$



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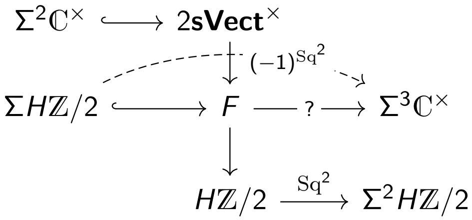

Question

What is the precise relation between such cocycles descriptions & the Postnikov towers of these groupoids?

See [Arn] for an example of such k-invariant calculations.

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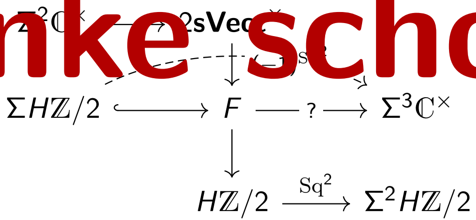
danke schön!



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References I

[Arn] Bertram Arnold, *Maps into a Postnikov tower of the sphere*. URL:https://mathoverflow.net/q/322250 (version: 2019-02-01).

[DHJF+24] Thibault D. Décoppet, Peter Huston, Theo Johnson-Freyd, Dmitri Nikshych, David Penneys, Julia Plavnik, David Reutter, and Matthew Yu, *The classification of fusion 2-categories*, arXiv, 2024.

[GJF19] Davide Gaiotto and Theo Johnson-Freyd, *Symmetry protected topological phases and generalized cohomology*, Journal of High Energy Physics **2019** (May 2019), no. 5.

[GW14] Zheng-Cheng Gu and Xiao-Gang Wen, *Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ-models and a special group supercohomology theory*, Physical Review B **90** (September 2014), no. 11.

[JFR23] Theo Johnson-Freyd and David Reutter, *Minimal nondegenerate extensions*, Journal of the American Mathematical Society **37** (July 2023), no. 1, 81–150.

[Ker24] David Kern, *Categorical spectra as pointed $(\infty, \mathbb{Z})$-categories*, arXiv, 2024.

References II

[LH23] Félix Loubaton and Simon Henry, *An inductive model structure for strict ∞-categories*, arXiv, 2023.

[Lou24] Félix Loubaton, *Categorical theory of (∞, ω)-categories*, arXiv, 2024.

[Mas] Naruki Masuda, *The algebra of categorical spectra*, Ph.D. Thesis, Johns Hopkins University.

[Reu21] David Reutter, *On the categorical spectrum of topological quantum field theories*, 2021. URL: `https://youtu.be/7jdyjLjp3BA`.

[SP] Chris Schommer-Pries, *Is there an accepted definition of (∞, ∞) category?*. URL:https://mathoverflow.net/q/134099 (version: 2017-12-15).

[Ste] German Stefanich, *Higher quasicoherent sheaves*, Ph.D. Thesis, UC Berkley.

[TY25] D. T. and Matthew Yu, *Mutual influence of symmetries and topological field theories* (2025), available at `2507.06304`.