# mod 2 cohomology of EM spaces (Serre)

**Thm** $H^\bullet(K(\mathbb{Z}/2,n),\mathbb{Z}/2) = \mathbb{Z}/2[Sq^{i_1}\cdots Sq^{i_r}(u)]$

where

→ $u$ is a generator of $H^n(\underbrace{K(\mathbb{Z}/2,n)}_{\text{only } \pi_n=\mathbb{Z}/2},\mathbb{Z}/2) = \mathbb{Z}/2$

→ $i_1 - i_2 - \cdots - i_r < n$

## Why care?

## I) COHOMOLOGY OPERATIONS

- Unstable: $H^n(X;G) \xrightarrow{\theta_X} H^{n+q}(X;H)$ natural in $X$

$$\underbrace{\mathrm{Nat}(H^n(-,G),H^{n+q}(-,H))}_{\text{coh. ops.}} \overset{B}{\simeq} \mathrm{Nat}([-,K(G,n)],H^{n+q}(-,H))$$

$$\overset{\text{coh. classes}}{\overset{\mathcal{Y}}{\simeq}} H^{n+q}(K(G,n),H)$$

- Stable: $H^n(X;G) \xrightarrow{\theta_n} H^{n+q}(X,H)$ commuting w/suspension

$$\wr$$

$$\varprojlim(\cdots \to H^{n+q+1}(K(G,n+1),H) \to H^{n+q}(K(G,n),H) \to \cdots)$$

$$\wr \text{ Freudenthal}$$

$$H^{N+q}(K(G,N),H), \quad N>q$$

# (II) POSTNIKOV SYSTEMS

$$
\begin{tikzcd}
 & & & \vdots \arrow[d] & \\
K(\pi_4,4) \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & X_4 \arrow[d] & X \arrow[l] \\
\bullet \arrow[r, dash] & K(\pi_3,3) \arrow[r] \arrow[d] & X_3^{4.\ell\ell\,\pi_1} \arrow[r] \arrow[d] & X_3 \arrow[r, "\Gamma h"'] \arrow[d] & \bullet \arrow[d] \\
 & \bullet \arrow[r] & K(\pi_2,2) \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & BK(\pi_3,3) = K(\pi_3,4) \\
 & & \bullet \arrow[r] & K(\pi_1,1) = X_1 \arrow[d] & \\
 & & & \bullet &
\end{tikzcd}
$$

$\hookrightarrow \in H^4(X_2, \pi_3)$

$X = K(\pi_1,1) \cdot K(\pi_2,2) \cdots$
$\downarrow$
k-invariants

$K(\mathbb{Z}/p,1) = L^p$

p' is stable

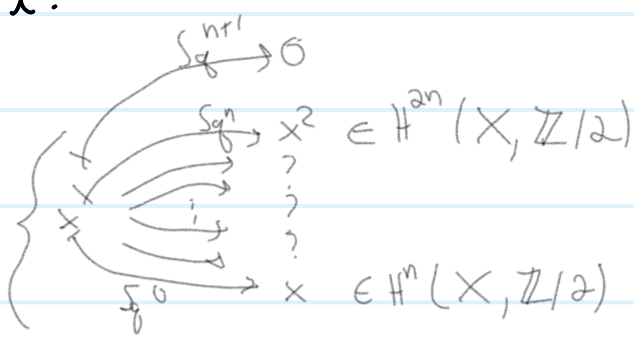
# (1) STEENROD SQUARES

for each $i \geq 0$

**Def** $Sq^i : H^n(X, \mathbb{Z}/2) \longrightarrow H^{n+i}(X, \mathbb{Z}/2)$

$\hookrightarrow$ cohomology operations s.t.

(1) $Sq^i(x) = \begin{cases} 0, & \deg(x) < i \\ x^2, & \deg(x) = i \\ x, & 0 = i \end{cases}$




(2) $Sq^i$ are stable

$\hookrightarrow$ commutes w/suspension

(3) etc.

**Def** An *iterated Steenrod square* is a composite $Sq^{i_1}\circ\cdots\circ Sq^{i_r} =: Sq^I$; $\rightsquigarrow e(I)=1$

This is *admissible* if $i_n \geq 2i_{n+1}$. $Sq^4 Sq^2 Sq^1$ ✓ $Sq^3 Sq^2$ ×

The *excess* of an admissible $Sq^I$ is the number

$$e(I) = i_1 - i_2 - \cdots - i_r \qquad Sq^5 Sq^2 Sq^1 \rightsquigarrow e(I)=2$$

$$= (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_{r-1} - 2i_r) + i_r \geq 0$$

We can understand the statement of the theorem: $e(I)=0 \Leftrightarrow Sq^0$

**Thm** $H^*(K(\mathbb{Z}/2,n),\mathbb{Z}/2) = \mathbb{Z}/2[Sq^I(u)]$

where
- $u$ is a generator of $H^n(K(\mathbb{Z}/2,n),\mathbb{Z}/2)$ $= \mathbb{Z}/2$ (Hurewicz)
- $I$ ranges over admissible sequences with $e(I) < n$

$Sq^I(u) \in H^{i_1+\cdots+i_r+n}(K(\mathbb{Z}/2,n))$

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## ② TRANSGRESSION IN THE SERRE S.S.

A fibration $F \to X \to B$;

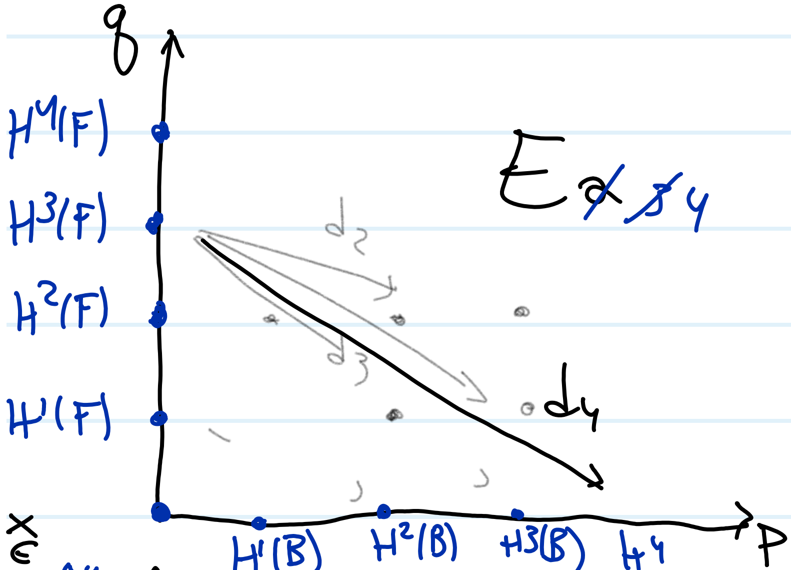
$$H^n(B) \longrightarrow H^n(F)$$

The *transgression* will be a partially defined map

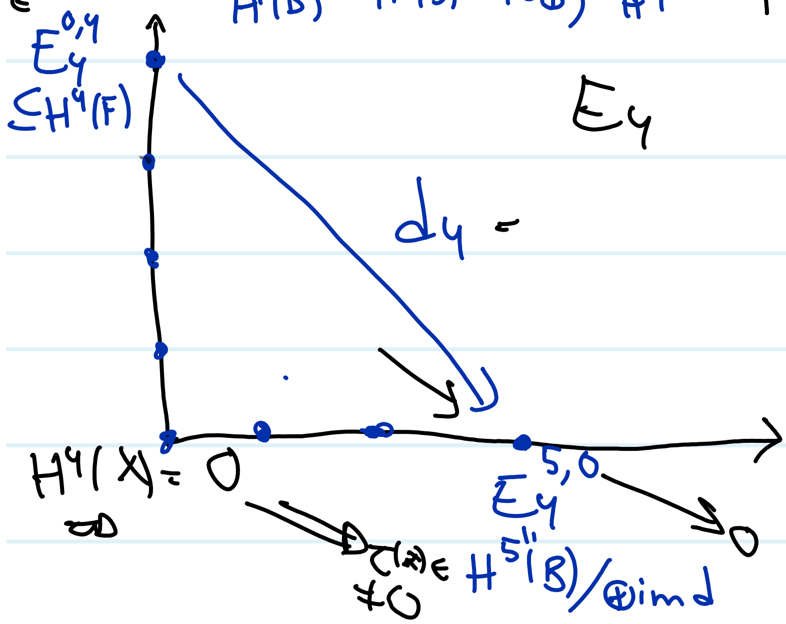
$$\tau \subseteq H^n(F) \longrightarrow H^{n+1}(B)/\sim$$

defined through the Serre spectral sequence of the fibration.

Suppose that $E_2^{p,q} = H^p(B) \otimes H^q(F)$




We turn the page by taking cohomology $E_{n+1}^{p,q} = \ker d / \operatorname{im} d$




As we turn the pages, we occasionally reach the "last" differential from the y- to the x-axis.

This is the transgression.

We want to understand the domain/codomain of $\tau$:

- DOMAIN: there is no image in $\ker d/\operatorname{im} d$, so to turn the page is to take kernels.

$\boxed{\text{Def}}$ $x \in E_2^{p,q}$ is *transgressive* if $d_3 x = \dots = d_{n-1} x = 0$.

- CODOMAIN: everything is in the kernel in $\ker d/\operatorname{im} d$, so to turn the page is to take a quotient $H^n(B)/\operatorname{im} d$.

$\boxed{\text{Def}}$ The *image* of a transgressive $x \in H^n(F)$ is any $y \in \tau(x)$.