Algebras and bimodules

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This is an exposition on the symmetric bicategory $\mathbf{Mor}_R$ of algebras, bimodules and bimodule homomorphisms, where R is commutative.

When $R = \mathbb{Z}$ this is the more common bicategory of rings, bimodules over them and bimodule homomorphisms.

1 Bicategories

We recall the definition of a bicategory and sketch that of a symmetric monoidal bicategory.

Definition 1. A **bicategory** $\mathcal{B}$ consists of

- a collection of *objects* $\mathrm{ob}(\mathcal{B})$;
- for each $x, y \in \mathrm{ob}(\mathcal{B})$, a *hom-category* $\mathcal{B}(x, y)$;
- for each $x, y, z \in \mathrm{ob}(\mathcal{B})$, a *composition functor*

$$\star : \mathcal{B}(y, z) \times \mathcal{B}(x, y) \to \mathcal{B}(x, z);$$

- for each $x \in \mathcal{B}(x, y)$, an *identity functor* $1_x : \{*\} \to \mathcal{B}(x, x)$.

Associativity and unitality aren't held on the nose. Instead there are the following natural isomorphisms:

- *Associators:*

$$\begin{tikzcd} \mathcal{B}(z,w) \otimes_R \mathcal{B}(y,z) \otimes_R \mathcal{B}(x,y) \arrow[r, "\mathrm{id}\otimes_R\star"] \arrow[d, "\star\otimes_R\mathrm{id}"'] & \mathcal{B}(z,w) \otimes_R \mathcal{B}(x,z) \arrow[d, "\star"] \arrow[dl, phantom, "\Leftarrow \alpha"] \\ \mathcal{B}(y,w) \otimes_R \mathcal{B}(x,y) \arrow[r, "\star"'] & \mathcal{B}(x,w) \end{tikzcd}$$

- *Unitors:*

$$\begin{tikzcd} \mathcal{B}(x,y) \otimes_R \{*\} \arrow[r, "\mathrm{id}\otimes_R 1_y"] \arrow[dr, bend right, "\cong"'] & \mathcal{B}(x,y) \otimes_R \mathcal{B}(y,y) \arrow[d, "\star"] \arrow[dl, phantom, "\Leftarrow \lambda"] \\ & \mathcal{B}(x,y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{B}(x,x) \otimes_R \mathcal{B}(x,y) \arrow[d, "\star"'] \arrow[dr, phantom, "\Rightarrow \rho"] & \{*\} \otimes_R \mathcal{B}(x,y) \arrow[l, "1_x\otimes_R\mathrm{id}"'] \arrow[dl, bend left, "\cong"] \\ \mathcal{B}(x,y) & \end{tikzcd}$$

Furthermore the associators and unitors are assumed to be coherent. This is equivalent to assume the pentagon and triangle identities hold pointwise.

The word "bicategory" usually invokes the picture of a globular 2-cell $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & y \end{tikzcd}$.

Here the morphisms and 2-cells here are respectively the objects and morphisms of $\mathcal{B}(x, y)$.

Ne can vertically compose 2-cells by using the composition of $\mathcal{B}(x,y)$:

$$
\begin{tikzcd}
x \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & y
\arrow[from=F, to=G, Rightarrow, "\eta"]
\arrow[from=G, to=H, Rightarrow, "\varepsilon"]
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
x \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "h"'{name=H}] & y
\arrow[from=F, to=H, Rightarrow, "\varepsilon\circ\eta"]
\end{tikzcd}
$$

Mertical composition is associative because it's the composition of the ordinary category $\mathcal{B}(x,y)$. So any order for the vertical composite of $\begin{tikzcd} \bullet \arrow[r, bend left=70, ""{name=A}] \arrow[r, bend left=20, ""{name=B}] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=20, ""{name=C}] \arrow[r, bend right=70, ""{name=D}] & \bullet \arrow[from=A, to=B, Rightarrow] \arrow[from=C, to=D, Rightarrow] \end{tikzcd}$ results in the same 2-cell.

The composition functor $\star$ yields a horizontal composition:

$$
\begin{tikzcd}
x \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "f'"'{name=F2}] & y \arrow[r, bend left=50, "g"{name=G}] \arrow[r, bend right=50, "g'"'{name=G2}] & z
\arrow[from=F, to=F2, Rightarrow, "\eta"]
\arrow[from=G, to=G2, Rightarrow, "\varepsilon"]
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
x \arrow[r, bend left=50, "g\star f"{name=F}] \arrow[r, bend right=50, "g'\star f'"'{name=H}] & z
\arrow[from=F, to=H, Rightarrow, "\varepsilon\star\eta"]
\end{tikzcd}
$$

The difference between bicategories and strict 2-categories is here: the composite of a sequence of 2-cells $\begin{tikzcd} x_0 \arrow[r, bend left, ""{name=A}] \arrow[r, bend right, ""'{name=B}] & x_1 \arrow[r, dotted, bend left] \arrow[r, dotted, bend right] \arrow[r, phantom, "\cdots"] & x_{n-1} \arrow[r, bend left, ""{name=C}] \arrow[r, bend right, ""'{name=D}] & x_n \arrow[from=A, to=B, Rightarrow] \arrow[from=C, to=D, Rightarrow] \end{tikzcd}$ no longer results in a unique 2-cell in $\mathcal{B}(x_0, x_n)$, but instead the different possibilities can be uniquely compared via the associators.

Example 2. The **delooping** of a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is the bicategory $\mathbf{B}\mathcal{M}$ defined by

- a single object $\bullet$;
- $\mathcal{M}$ as the unique hom-category $\mathbf{B}\mathcal{M}(\bullet,\bullet)$;
- $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ as the composition;
- $I : \{*\} \to \mathcal{M}$ as the identity;
- α, λ and ρ as the associators and unitors.

This isn't a strict 2-category unless $\mathcal{M}$ is a strict monoidal category (because $(x \otimes y) \otimes z \neq x \otimes (y \otimes z)$). Additionally, note that this construction identifies monoidal categories as the special case of bicategories with a single object, for then we may extract a monoidal category from the unique hom-category.

Example 3. There is a bicategory $\mathbf{Mor}_{\mathbb{Z}}$ of rings, bimodules and bimodule homomorphisms. The vertical composition is just the composition of bimodule homomorphisms, and the horizontal composition is given by tensoring over the middle ring:

$$R \xrightarrow{M} S \xrightarrow{N} T \;\mapsto\; R \xrightarrow{M \otimes_S N} T \;.$$

This the prototype for $\mathbf{Mor}_R$.

Definition 4. An **adjunction** between two morphisms$f : x \to y$ and $g : y \to x$ in a bicategory $\mathcal{B}$ is a pair of 2-cells

$$
\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, equal] \arrow[rr, phantom, bend left=20, "\Uparrow\eta"] & & x
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
y \arrow[rr, equal] \arrow[dr, "g"'] & & y \\
& x \arrow[ur, "f"'] \arrow[u, phantom, "\varepsilon\Uparrow"] &
\end{tikzcd}
\quad .
$$

such that the *triangle identities* hold:

$$
\begin{tikzcd}
& y \arrow[rr, equal] \arrow[dr, "g"] & & y \\
x \arrow[ur, "f"] \arrow[rr, equal] \arrow[rr, phantom, bend left=20, "\eta\Uparrow"] & & x \arrow[ur, "f"'] \arrow[u, phantom, "\Uparrow\varepsilon"] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& y \arrow[r, equal] & y \\
x \arrow[ur, "f"] \arrow[r, equal] & x \arrow[ur, "f"'] \arrow[u, phantom, "1_f"] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
y \arrow[rr, equal] \arrow[dr, "g"'] & \arrow[d, phantom, "\Uparrow\varepsilon"] & y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\eta"] & \\
& x \arrow[ur, "f"] \arrow[rr, equal] & \, & x
\end{tikzcd}
=
\begin{tikzcd}
y \arrow[r, equal] \arrow[dr, "g"'] & y \arrow[dr, "g"] \arrow[d, phantom, "1_g"] & \\
& x \arrow[r, equal] & x
\end{tikzcd}
.
$$

Example 5. An object x in a monoidal category $\mathcal{M}$ is dualizable precisely if it has an adjoint when regarded as a morphism in $\mathbf{B}\mathcal{M}$. A left adjoint is a left dual.

Definition 6. An **equivalence** between two objects x and y in a bicategory $\mathcal{B}$ is a pair of morphisms $f : x \to y$ and $g : y \to x$ and 2-isomorphisms $\eta : 1_x \cong gf$ and $\varepsilon : fg \cong 1_y$.

Example 7. Two rings R and S are equivalent in $\mathbf{Mor}_{\mathbb{Z}}$ if there is an (R, S)-bimodule M and a (S, R)-bimodule N such that

$$N \otimes_S M \cong S \text{ and } M \otimes_R N \cong R. \tag{1}$$

A theorem in Morita theory implies that this condition is precisely equivalent to requiring R and S to be Morita equivalent, i.e. possessing equivalent categories of left-modules. See the first page of [Mey] for a brief discussion about this.[1]

Symmetric bicategories

A **monoidal bicategory** is a one-object tricategory [GPS95, Definition 2.6.(ii)]. Another way of putting this is that a monoidal bicategory consists of a bicategory $\mathcal{B}$ equipped with a pair of pseudofunctors

$$\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \text{ and } I : \{*\} \to \mathcal{B}$$

plus additional coherence data. A *braided* monoidal category additionally posseses equivalences

$$b_{xy} : x \otimes y \to y \otimes x$$

and further coherence data. *Symmetric* bicategories have an extra modification and another coherence axiom.
Alas, we won't delve into the details as they are famously messy and long. Instead we refer the reader to the appendix of [SP11] for details and some nice pictures.
Another interesting point of view if found in [Shu10], where the strategy is to obtain symmetric bicategories from symmetric *double* categories, and the axioms for these are simpler. The main symmetric bicategory of this text can be constructed in this way.

2 Bimodules and their homomorphisms

In this section we define bimodules and bimodule homomorphisms over R-algebras, which we assume to be associative and unital i.e. monoid objects in $\mathbf{Mod}_R$.

Definition 8. Let (A, μ, I_A) be an R-algebra. A **left A-module** is an R-module (M, ρ) equipped with a linear map

$$m : A \otimes_R M \to M$$

which is unital and associative with respect to the multiplication in A:

$$
\begin{tikzcd}
A \otimes_R A \otimes_R M \arrow[r, "\mathrm{id}_A \otimes_R \rho"] \arrow[d, "\mu \otimes_R \mathrm{id}_M"'] & A \otimes_R M \arrow[d, "\rho"] \\
A \otimes_R M \arrow[r, "\mu"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes M \arrow[r, "I_A \otimes_R \mathrm{id}_M"] \arrow[dr, "m"'] & A \otimes_R M \arrow[d, "m"] \\
& M
\end{tikzcd}
$$

[1] The first theorem in that paper implies that a Morita equivalence between $T : \mathbf{Mod}_R \cong \mathbf{Mod}_S$ is given by $T \cong N \otimes_S (-)$, where N is a (S, R)-bimodule. In particular there must be an (R, S)-bimodule N such that

$$N \otimes_S \otimes M \otimes_R (-) \cong \mathrm{id}_{\mathbf{Mor}_R} \text{ and } M \otimes_N \otimes P \otimes_S (-) \cong \mathrm{id}_{\mathbf{Mor}_S}.$$

If we are given M and N such that Equation 1 holds, then it's immediate that they will form an equivalence such as this one. Conversely, given such an equivalence we recover Equation 1 by plugging in the identities for R and S.

Right modules are defined similarly.

Remark 9. Actually, right modules are defined dually. To see that, define the opposite algebra A^{op} to be A equipped with the same identity and multiplication given by

$$a \cdot^{\mathrm{op}} b := b \cdot a.$$

(This makes sense in any symmetric monoidal category.)

In particular note that if A is commutative then it coincides with A^{op} and the notions of left and right module coincide.

Definition 10. An **(A,B)-bimodule** is an R-module M equipped which is

- a left A-module $\rho_A : A \otimes_R M \to M$
- a right B-module $\rho_B : M \otimes B \to M$

and such that

$$\begin{tikzcd} A \otimes_R M \otimes_R B \arrow[r, "\mathrm{id}_A \otimes \rho_B"] \arrow[d, "\rho_A \otimes \mathrm{id}_B"'] & A \otimes_R M \arrow[d, "\rho_A"] \\ M \otimes_R B \arrow[r, "\rho_B"'] & M \end{tikzcd} .$$

In this case we write $M : A \nrightarrow B$.

Definition 11. A **left module homomorphism** between a pair of left A-modules M, N is a linear map $f : M \to N$ such that

$$\begin{tikzcd} A \otimes_R M \arrow[r, "m"] \arrow[d, "\mathrm{id}_A \otimes f"'] & M \arrow[d, "f"] \\ A \otimes_R M \arrow[r, "n"'] & N \end{tikzcd} .$$

Right module homomorphisms are defined dually, and a bimodule homomorphism consists of a pair of left and right homomorphisms.

The composition of bimodule homomorphisms is associative and the identity of $M : A \nrightarrow B$ is id_M. So there is a category $\mathbf{Mor}_R(A, B)$.

3 The bicategory of bimodules

The main piece remaining for $\mathbf{Mor}_R$ is the horizontal composition. We will construct this by the *balanced tensor product*

$$\boxtimes_B : \mathbf{Mor}_R(A, B) \times \mathbf{Mor}_R(B, C) \to \mathbf{Mor}_R(A, C).$$

We first set up the balanced tensor product of bimodules, which takes $N : B \nrightarrow C$ and $M : A \nrightarrow B$ and yields an bimodule $M \boxtimes_B N : A \nrightarrow C$. The construction is akin to the tensor product of vector spaces.

The tensor product of vector spaces

Given two $\mathbb{K}$-vector spaces V and W, their tensor product is defined as a $\mathbb{K}$-vector space $V \otimes_{\mathbb{K}} W$ such that bilinear maps $V \otimes_R W \to U$ correspond uniquely to linear maps $V \otimes_{\mathbb{K}} W \to U$:

$$\begin{tikzcd} V \otimes_R W \arrow[r, "\otimes"] \arrow[dr, "\forall f"'] & V \otimes_{\mathbb{K}} W \arrow[d, dashed, "\exists! \tilde{f}"] \\ & U \end{tikzcd} .$$

In other words, consider the functor

$$\mathrm{Bilin}(V, W; -) : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Set}$$

taking a vector space to the set of all bilinear maps $V \otimes_R N \to U$ The tensor product is a representative to this functor:

$$\mathrm{Bilin}(V, W; -) \cong \mathbf{Vect}_{\mathbb{K}}(V \otimes_{\mathbb{K}} W; -).$$

Definition 12. Let M and N be respectively right and left B-modules. A **B-balanced map** is an R-bilinear map $f : M \otimes_R N \to U$, where U is an R-module, such that

$$f(m \otimes b, n) = f(m, b \otimes n)$$

for all m in M, n in N and a in A.

Example 13. If R is a field then a balanced map is a bilinear map.

Given M and N such as above, notice that there is a functor

$$\mathrm{Bal}(M, N; -) : \mathbf{Mod}_R \to \mathbf{Set}$$

taking an R-module U to the set of R-balanced maps $M \otimes_R N \to U$.

Definition 14. The **balanced tensor product** $M \boxtimes_A N$ between a right A-module M and a left A-module N is a representative for $\mathrm{Bal}(M, N; -)$.

In terms of a universal property:

$$\begin{tikzcd} M \otimes_R N \arrow[r, "\boxtimes"] \arrow[dr, "\forall f"'] & M \boxtimes_A N \arrow[d, dashed, "\exists! \tilde{f}"] \\ & U \end{tikzcd} .$$

Remark 15. By the Yoneda embedding the balanced tensor product is defined up to isomorphism - *if it exists*. This is the case: an explicit construction is obtained by taking the quotient of $M \otimes_R N$ by the relations

$$(m \cdot a) \otimes_R b \sim m \otimes_R (an)$$

for all a in A.

Proposition 16. *Let M be a right A-module and N and P be left A-modules, and suppose there is a left-module homomorphism $f : N \to P$. Then there is a canonical morphism*

$$id_M \boxtimes_A f : M \boxtimes_A N \to M \boxtimes_A P.$$

A similar statement holds for right modules.

Proof. This is induced by the diagram

$$\begin{tikzcd} M \otimes_R N \arrow[r, "\mathrm{id}_M \otimes_R f"] \arrow[d, "\boxtimes"'] & M \otimes_R P \arrow[d, "\boxtimes"] \\ M \boxtimes_A N \arrow[r, dashed, "\mathrm{id}_M \otimes f"'] & M \boxtimes_A P \end{tikzcd}$$

where $\boxtimes \circ (\mathrm{id}_M \otimes_R f)$ is A-balanced by construction:

$$\begin{aligned} \boxtimes \circ (\mathrm{id}_M \otimes_R f)(mb \otimes n) &= \boxtimes(mb \otimes_R f(n)) \\ &= \boxtimes(m \otimes_R bf(n)) \\ &= \boxtimes(m \otimes_R f(bn)) \\ &= \boxtimes \circ (\mathrm{id}_M \otimes_R f)(m \otimes bn) \end{aligned}$$

□

Corollary 17. *Given an A-module M defines a functor, the construction above defines a functor*

$$M \boxtimes_A (-) : {}_A\mathbf{Mod} \to \mathbf{Mod}_R.$$

In other words, given right A-module homomorphisms $N \to P$ and $P \to Q$ we have

$$(id_M \otimes g) \circ (id_M \otimes f) = id_M \otimes (g \circ f) \text{ and } id_M \otimes id_N = id_{M \boxtimes_A N}.$$

Proof. Preservation of composition follows from the diagram below because of the uniqueness of the induced maps.

$$\begin{tikzcd} M \otimes_R N \arrow[r, "\mathrm{id}_M \otimes_R f"'] \arrow[d, "\otimes"'] \arrow[rr, bend left, "\mathrm{id}_M \otimes_R (g \circ f)"] & M \otimes_R P \arrow[r, "\mathrm{id}_M \otimes_R g"'] \arrow[d, "\otimes"] & M \otimes_R Q \arrow[d] \\ M \boxtimes_A N \arrow[r, dashed, "\mathrm{id}_M \otimes f"] \arrow[rr, dashed, bend right, "\mathrm{id}_M \otimes (g \circ f)"'] & M \boxtimes_A P \arrow[r, dashed, "\mathrm{id}_M \otimes g"] & M \boxtimes_B Q \end{tikzcd}$$

Preservation of identities is similar. □

Corollary 18. *Let M and N be respectively right and left B-module. Then*

- *if M is a left A-module, then $M \boxtimes_B N$ is an A-left module;*
- *if N is a right C-module then $M \boxtimes_B N$ is an C-right module.*

Proof. We prove the second statement: for any $c \in C$ let $\mu_c : N \to N$ be the automorphism given by $\mu_c(n) = n \cdot c$. Then the right C-module structure on $M \boxtimes_B N$ is obtained by defining the action of $c \in C$ to be $\mathrm{id}_M \otimes \mu_c$. Compatibility with scalars follows from functoriality and the definition of bimodule.

The other case is analogous. □

Corollary 19. *The balanced tensor product of $M : A \nrightarrow B$ and $N : B \nrightarrow C$ is a bimodule*

$$M \boxtimes_B N : A \nrightarrow C.$$

Proposition 20. *Given bimodules $M :\nrightarrow B$, $N : B \nrightarrow C$ and $P : C \nrightarrow D$, there is a unique isomorphism of (A, D)-modules*

$$(M \boxtimes_A N) \boxtimes_B P \cong M \boxtimes_A (N \boxtimes_B P)$$

Proof. The isomorphism in all instances is obtained by unravelling the universal properties involved and the definition in Corollary 18. □

Proposition 21. *The balanced tensor product is unital with identity given by the algebra itself, i.e. if M is a left A-module then*

$$A \boxtimes_A M \cong M$$

where A is regarded a right A-module over itself. A similar statement hold for right-modules.

Proof. This only happens because A is unital, which allows us to find a section to the multiplication $\rho : M \otimes_R A \to M$ by

$$s : M \cong M \otimes_R R \xrightarrow{\mathrm{id}_M \otimes 1_A} M \otimes_R A.$$

Then the result follows from the universal property. $\square$

Definition 22. Consider the following situation:

$$\begin{tikzcd} A \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow f"] & B \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow g"] & C \end{tikzcd} .$$

In this case, the tensor product of f and g

$$g \boxtimes_B f : M \boxtimes_B N \to M' \boxtimes_B N'$$

is the map induced by functoriality (see Corollary 17). It is a homomorphism by definition.

Definition 23. For any ring R, let $\mathbf{Mor}_R$ be the bicategory whose

- objects are R-algebras;
- hom-categories are the categories $\mathbf{Mor}_R(A, B)$ defined in Section 2;
- composition functors

  $$\star : \mathbf{Mor}_R(A, B) \otimes_R \mathbf{Mor}_R(B, C) \to \mathbf{Mor}_R(A, C)$$

  are given by the balanced tensor product of right and left B-modules;
- identities are given by A seen as an (A, A)-bimodule with id_A as the identity 2-cell.

The associators and unitors are given by the unique isomorphisms described above.

The monoidal structure of $\mathbf{Mod}_R$

The tensor product of bimodules $M : A \nrightarrow B$ and $M' : A' \nrightarrow B'$ can be defined simply as their R-tensor product. For instance, a left $(A \otimes_R A')$-module structure is induced on $M \otimes_R M'$ because

$$(A \otimes_R A') \otimes_R M \otimes_R M' \cong (A \otimes_R M) \otimes_R (A' \otimes_R M'),$$

and from here we can just take the tensor product of the scalar multiplications. This construction extends similarly to bimodule homomorphisms.

References

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