THESIS PROPOSAL

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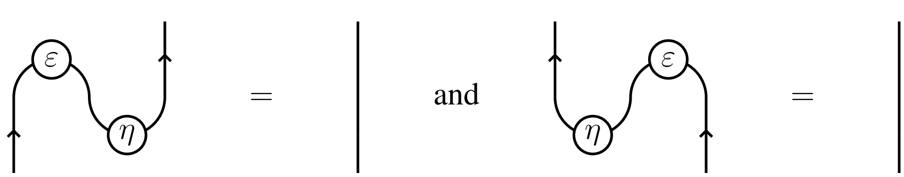
CONTENTS

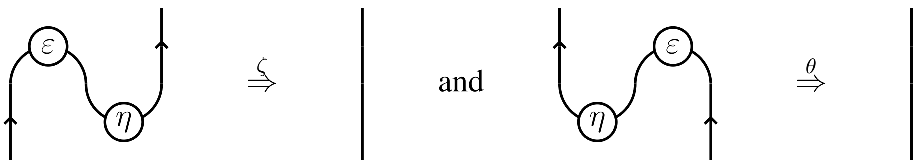
1. INTRODUCTION

Recall that a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is *compact* if every object has a dual. The dual of an object v is exhibited by another object v^R and morphisms $\eta : \mathbf{1} \to v \otimes v^R$ and $\varepsilon : v^R \otimes v \to \mathbf{1}$ satisfying the triangle identities:

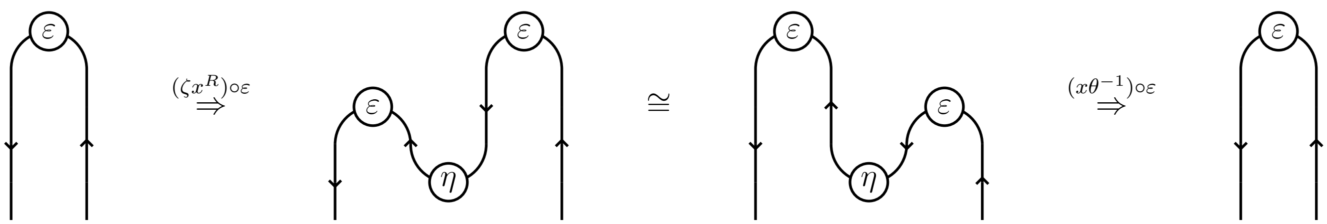


This definition is eerily similar to right adjoints, and in fact those concepts coincide: in a bicategory $\mathcal{B}$, a right adjoint of a morphism $f : x \to y$ is exhibited by another morphism $f^R : y \to x$ and 2-morphisms $\eta : 1_y \to f \otimes f^R$ and $\varepsilon : f^R \otimes f \to 1_x$ satisfying the same triangle identities. Then right dualizability in $\mathcal{C}$ coincides with right adjunctability in the delooping $B\mathcal{C}$.

However when $(\mathcal{B}, \otimes, \mathbf{1})$ is a *symmetric bi*category, the right notion of dualizability replaces the triangle identities with iso-2-morphisms



required to satisfy the *swallowtail equation* [S16, pp. 775-776], which says that the following composition is an identity 2-morphism:



The symmetric bicategory is *compact* if every object has a dual in this sense. Some examples:

- The bicategory Span$(\mathcal{C})$ of spans in a complete category, where the tensor product is given by the categorical product.
- The bicategory Prof of categories, profunctors and natural transformations, where the tensor product is given by the product of categories.
- The bicategory $\mathrm{Cob}_{d,d-1,d-2}$ of d-dimensional cobordisms is expected to be compact, although a formal proof has only appeared for $d = 2$ in [SP11].[1]

We could also ask if ε and η are themselves two-sided adjoints. If so, we say that x is *2-dualizable*; if all objects of $\mathcal{B}$ are 2-dualizable, we say that $\mathcal{B}$ is *fully dualizable*.

- If $\mathcal{B}$ is a fully dualizable bicategory, the *cobordism hypothesis* asserts that there is the pseudofunctors $\mathrm{Cob}_{2,1,0} \to \mathcal{B}$ are classified by the objects of $\mathcal{B}$ by taking the image of the point [BD95, L09a] .

Note that the 2-dualizability of a dualizable object x is answered by studying the bicategory $\mathcal{B}$ (with no tensor product).

Now, here is a punchline: *the swallowtail equation is redundant*! By this we mean that even if ζ and θ witness a non-coherent dual, in the sense that they violate the swallowtail equation, they still give rise to a coherent one. In other words, even the 1-dualizability of $x \in \mathcal{B}$ is answered in terms of the (ordinary!) symmetric category $h_1(\mathcal{B})$ obtained by quotienting $\mathcal{B}$ by iso-2-morphisms.

[1]The symmetric bicategory $\mathrm{Cob}_{d,d-1,d-2}$ could be obtained from the corresponding symmetric $(\infty, 2)$-category, but as far as I am aware this constructions isn't yet available. In fact, even the non-symmetric is still to soon appear in the literature [R23].

In fact, this is a statement about adjunctions in tricategories, i.e. that a non-coherent adjunction in a tricategory gives rises to a coherent one.[2] In other words, to check the adjunctability of a morphism $f \in \mathcal{C}$ it suffices to do that in the bicategory $h_2(\mathcal{C})$.

Similarly, we could also ask if the 2-morphisms ε and η of an adjunction $f \dashv f^R$ are themselves adjoints. If so, we say that f is *2-adjunctable*; if all morphisms of $\mathcal{C}$ are 2-adjunctable, then $\mathcal{C}$ if *fully adjunctable*. This question can be answered by studying the bicategory $\mathrm{Mor}(\mathcal{C})$ of morphisms, 2-morphisms and 3-morphisms of $\mathcal{C}$.

These facts generalize to weak n-categories:

- An object x in a symmetric weak n-category is fully dualizable if it is dualizable and the 1-morphisms η and ε are adjunctable, and so on.
- A non-coherent adjunction in a weak $(n+1)$-category can be promoted to a coherent adjunction. In particular, for symmetric weak n-categories non-coherent duals can be made coherent.
- *The full adjunctability of a weak n-category is studied in terms of the bicategories*

$$h_2(\mathcal{C}), h_2(\mathrm{Mor}(\mathcal{C})), \dots h_2(\mathrm{Mor}^{n-3}(\mathcal{C})), \mathrm{Mor}^{n-2}(\mathcal{C})$$

Some details to be ensued, since in the current technology n-categories are dealt as (∞, n)-categories. For instance, the main result in [RV16, Section 4] shows that adjunctions in quasi-categorically enriched categories can be extended to homotopy coherent adjunctions. Since these model $(\infty, 2)$-categories [L09c], we can interpret the statement as a "swallowtail equation" coherence theorem. See also [B16] and discussion therein.

Writing this sequence of bicategories as

$$\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{n-2}, \mathcal{B}_{n-1},$$

we observe that it is characterized by the property that $h_1(\mathcal{B}_k) = \mathrm{Mor}(\mathcal{B}_{k-1})$. Then the last remark above is that such sequences of bicategories are enough to study questions regarding about full dualizability or adjunctability.

If we arrange the bicategories in the sequence as follows, they look like a series of rungs. So we call them *ladders of bicategories*.

$$\begin{array}{cccc}
n & \mathcal{B}_2 & & \\
4 & \mathbb{H} & \mathcal{B}_1 & \\
3 & H & [H] & \mathcal{B}_0 \\
2 & \eta & \eta & [\eta] \\
1 & & f & f \\
0 & & & x
\end{array}$$

The main goal of this thesis project is to pin down appropriate definitions for ladders, and construct a categorical ambient to study them. Here are some questions that we have in mind:

[2] Here is a proof in Globular: `http://globular.science/1512.006`. It is due to Jamie Vicary.

- What is the correct ambient category for ladders? What properties does it have?
- How does the homotopy theory of ladders interact with the current landscape of higher categories?
- Questions about full dualizability can be answered in terms of ladders. What else?

Throughout this note we sketch a partial answer to some of this questions, and

Organization of this note. In Section 2 we do some preliminary work regarding strict n-categories. Then, from Sections 3 to 7, we explain the current state of the project, which involves various ideas regarding ladders of different flavours. We study some of their categorical properties such as accessibility. In Section 8 we discuss some remaining problems related to the ideas developed throughout the note, and we point out to other directions related to the main idea of the project.

There are two appendices. In the first, we fill the details of a folkloric proof of the canonical model structure for categories and 2-categories. In the second, we review background material on gaunt n-categories.

2. Homotopy 2-categories of strict n-categories

In this section we recall the inductive construction of strict n-categories and how to extract their homotopy 2-categories. When we collect strict n-categories in a single category, we are always referring to small ones.

Recall that if $\mathcal{V}$ is a symmetric monoidal category then so is the category $\mathcal{V}\,\mathrm{Cat}$ of categories enriched over $\mathcal{V}$. Starting with the category Set of sets, this leads to the following inductive definition.

Definition 2.1. A locally small **strict n-category** is a category enriched over small strict $(n-1)$-categories. A strict n-category with a set of objects is called **small**, and we denote the category that they form by Cat_n.

We reserve the term "n-category" to talk about strict n-categories. We will always specify otherwise, either by referring to weak n-categories or to (∞, n)-categories. A bicategory is, of course, a weak 2-category.

Example 2.2. The categories Cat (resp. Cat_2) consists of the small categories (resp. small 2-categories).

Example 2.3. The **suspension** of an n-category $\mathcal{C}$ is the $(n+1)$-category $\Sigma\mathcal{C}$ whose set of objects is $\{\bot, \top\}$ and

$$\Sigma\mathcal{C}(x,y) = \begin{cases} * & \text{if } x = y \\ \mathcal{C} & \text{if } x = \bot \text{ and } y = \top \\ \varnothing & \text{otherwise} \end{cases}.$$

The suspension defines a functor $\Sigma : \mathrm{Cat}_n \to \mathrm{Cat}_{n+1}$ that is not cocontinuous[3] However, if we change its target to the category $\mathrm{Cat}_{n+1}^{\{0,1\}}$ of *bipointed* $(n+1)$-categories, a right adjoint is obtained by sending $\mathcal{C}_{x_0,x_1}$ to $\mathcal{C}(x_0, x_1)$.

[3] It doesn't preserve initial objects.

Recall that a category is κ-*accessible* if it has κ-filtered colimits and a set of κ-compact objects that generate the category under filtered colimits. A cocomplete κ-accessible category is called κ-*locally presentable*. If $\kappa = \aleph_0$, the category is called *locally finitely presentable*.

Lemma 2.4. *The category* Cat_n *is finitely locally presentable.*

Proof. If $\mathcal{V}$ is κ-locally presentable then so is $\mathcal{V}\,\mathrm{Cat}$ [K01]. Since Set is locally finitely presentable, the result follows by induction. An explicit finite set of generators is the walking morphism $\bullet \to \bullet$ and its suspensions (see [BSP21, Proposition 2.7]). $\square$

Within an n-category $\mathcal{C}$, an *isomorphism* between objects x and y consists of an object f in $\mathcal{C}(x, y)$ and an object g in $\mathcal{C}(y, x)$ such that $gf = 1_x$ and $fg = 1_y$.

Let $h_0(\mathcal{C})$ denote the set of isomorphism classes of objects of $\mathcal{C}$.

Proposition 2.5. *This construction defines a functor* $h_0 : \mathrm{Cat}_n \to \mathrm{Set}$.

Proof. Given an n-functor $F : \mathcal{C} \to \mathcal{D}$, the assignment $[x] \mapsto [Fx]$ is well-defined because F preserves isomorphisms, and functorial because F is functorial. $\square$

Lemma 2.6. *The functor* h_0 *preserves products and terminal objects.*

Proof. A morphism (f, g) in $\mathcal{C} \times \mathcal{D}$ is invertible if and only if both f and g are invertible, so $h_0(\mathcal{C} \times \mathcal{D})$ is $h_0(\mathcal{C}) \times h_0(\mathcal{D})$. Also, since the terminal object of Cat_n has only one object and identity n-morphisms it is sent to a set with one element. $\square$

Corollary 2.7. *The functor* h_0 *induces a functor* $h_1 : \mathrm{Cat}_{n+1} \to \mathrm{Cat}$.

Proof. It is well-known that a lax monoidal functor $F : \mathcal{V} \to \mathcal{W}$ induces a functor $F_* : Cat(\mathcal{V}) \to \mathrm{Cat}(\mathcal{W})$ (see the Appendix of [Tei22]). Hence the existence of h_1 as $(h_0)_*$ is guaranteed by the previous lemma. $\square$

Lemma 2.8. *The functor* h_1 *also preserves products and terminal objects.*

Proof. The product in Cat_n is given by the product of the sets of objects and the product of the hom-categories, from which we see that $h_1(\mathcal{C} \times \mathcal{D})$ is $h_1(\mathcal{C}) \times h_1(\mathcal{D})$. The statement for terminal objects is trivial. $\square$

Corollary 2.9. *The functor* h_1 *induces a functor* $h_2 : \mathrm{Cat}_n \to \mathrm{Cat}_2$.

Proof. Restrict h_1 to the small $(n-1)$-categories. $\square$

We call $h_2(\mathcal{C})$ the **homotopy 2-category** of $\mathcal{C}$, which has the same objects as $\mathcal{C}$ and its hom-categories are the homotopy 1-categories $h_1\mathcal{C}(x, y)$. In other words, its morphisms are also the morphisms of $\mathcal{C}$, but 2-morphisms are only considered up to 3-isomorphisms.

Although h_0 preserves arbitrary products and coproducts (including empty and infinite), this is generally false for arbitrary limits and colimits:

Proposition 2.10. *The functor* $h_0 : \mathrm{Cat}_n \to \mathrm{Set}$ *is neither continuous nor cocontinuous.*

Proof. It suffices to give a counterexample for $n = 1$ and suspend it $(n-1)$ times.

For continuity, let $\mathcal{G}$ be a connected groupoid and consider two distinct functors $* \rightrightarrows \mathcal{G}$. They are equal upon taking h_0, but have empty equalizer in Cat.

For cocontinuity, consider the category A freely generated by the graph $f : x \leftrightarrows y : g$, and the categories B obtained by identifying $gf = 1_x$, and C by identifying $fg = 1_y$. The pushout of $B \leftarrow A \to C$ is the walking isomorphism, but A, B and C each have two isomorphism classes, and the walking isomorphism has just one. ☐

Corollary 2.11. *The functors h_1 and h_2 are not continuous nor cocontinuous.*

Nevertheless, these functors are all *accessible*, meaning that their source and target are accessible and that they commute with filtered colimits.

Proposition 2.12. *The functor $h_0 : \mathrm{Cat} \to \mathrm{Set}$ is accessible.*

Proof. Note that h_0 can be decomposed as a right adjoint followed by a left adjoint,

$$
\begin{tikzcd}
\mathrm{Cat}_n \arrow[dr, shift left, "\mathcal{M}"] \arrow[dd, "h_0"'] & \\
& \mathrm{Gpd}_n \arrow[ul, hook, shift left] \arrow[ul, phantom, "\top"] \arrow[dl, shift right, "\pi_0"'] \arrow[dl, phantom, "\top"] \\
\mathrm{Set} \arrow[ur, shift right, "D"'] &
\end{tikzcd},
$$

where $\mathcal{M}$ sends an n-category to its maximal sub-n-groupoid,[4] and π_0 takes isomorphism classes.

Left adjoints are clearly accessible, and the adjoint functor theorem for locally presentable categories [AR94, Theorem 1.66] asserts right adjoints between locally presentable categories are accessible. Since accessible functors are closed under the composition [MP89], it follows that h_0 is accessible. ☐

Lemma 2.13. *If $F : \mathcal{V} \to \mathcal{W}$ is an accessible lax monoidal functor between accessible categories, then so is the induced functor $F_* : \mathcal{V}\,\mathrm{Cat} \to \mathcal{W}\,\mathrm{Cat}$.*

Proof. Since F_* acts as the identity on objects, it suffices to check that for any directed colimit $\operatorname*{colim}_{i\in I} f_i$ in $\mathcal{V}\,\mathrm{Cat}$ we have

$$\operatorname*{colim}_{i\in I} F(f_i(x,y)) = F(\operatorname*{colim}_{i\in I} f_i(x,y)).$$

This is immediate since F is accessible. ☐

Corollary 2.14. *The functors h_1 and h_2 are accessible.*

[4]The maximal sub-n-subgroupoid also can be defined inductively from $\mathcal{M} : \mathrm{Cat} \to \mathrm{Set}$.

Given a locally small $(n+1)$-category $\mathcal{C}$, let $\operatorname{Mor}(\mathcal{C})$ be the n-category given as the disjoint union of its hom-n-categories[5]

$$\operatorname{Mor}(\mathcal{C}) := \coprod_{x,y \in \operatorname{ob}(\mathcal{C})} \hom_{\mathcal{C}}(x,y).$$

Proposition 2.15. *This construction defines a right adjoint functor* $\operatorname{Mor} : \operatorname{Cat}_{n+1} \to \operatorname{Cat}_n$.

Proof. We will construct the left adjoint $L : \operatorname{Cat}_n \to \operatorname{Cat}_{n+1}$ explicitly. First note that an n-category can regarded as the disjoint union of its connected components. Then, given a small strict n-category $\mathcal{C}$ with connected components $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$, define the $L\mathcal{C}$ to be the disjoint union of the *suspension* of each $\mathcal{C}_i$ (see Example 2.3).

Then observe that a 2-functor $L\mathcal{C} \to \mathcal{D}$ consists of a pair of objects x_i, y_i in $\mathcal{D}$ for each component of $\mathcal{C}$, and a functor $F_i : \mathcal{C}_i \to \hom_{\mathcal{D}}(x_i, y_i)$. The disjoint union of all F_i assembles into a functor $F : \mathcal{C} \to \operatorname{Mor}(\mathcal{D})$, and vice-versa, exhibiting adjointness. $\square$

Corollary 2.16. *The functor* Mor *is accessible.*

We now define the higher homotopy 2-category functors $h_2^k : \operatorname{Cat}_n \to \operatorname{Cat}_2$, $0 \le k \le n-2$, as follows:

- $h_2^{(0)}$ is h_2;
- $h_2^{(1)}$ is $h_2 \circ \operatorname{Mor}$, and in general $h_2^{(k)}$ is $h_2 \circ \operatorname{Mor}^k$.

Example 2.17. The $h_2^{(k)}$ functors for Cat_5 are depicted below:

$$\begin{tikzcd}
\operatorname{Cat}_5 \arrow[r, "h_2"] & \operatorname{Cat}_2 \arrow[rrr, dashed, no head] & & & h_2^{(0)} \\
{} \arrow[r, "\operatorname{Mor}"] & \operatorname{Cat}_3 \arrow[r, "h_2"] & \operatorname{Cat}_2 \arrow[rr, dashed, no head] & & h_2^{(1)} \\
& {} \arrow[r, "\operatorname{Mor}"] & \operatorname{Cat}_3 \arrow[r, "h_2"] & \operatorname{Cat}_2 \arrow[r, dashed, no head] & h_2^{(2)} \\
& & {} \arrow[r, "\operatorname{Mor}"] & \operatorname{Cat}_2 \arrow[r, dashed, no head] & h_2^{(3)}
\end{tikzcd}$$

Note that each $h_2^{(k)}$ is accessible since they are the composite of accessible functors.

3. Ladders of strict 2-categories

In this section we define ladders of strict 2-categories and describe the canonical ladder of a strict n-category. In particular, we show that ladders form an accessible category Lad_n.

Definition 3.1. A **ladder of 2-categories** is a (possibly infinite) sequence of 2-categories $\mathcal{B} = (\mathcal{B}_0, \ldots, \mathcal{B}_n)$ such that $h_1\mathcal{B}_{k+1} = \operatorname{Mor}(\mathcal{B}_k)$ for $0 \le k \le n-1$.

[5]When $n = 1$ we have to restrict ourselves to small categories due to size issues.

The category of ladders with n entries Lad_n is obtained by observing that those ladders belong to the pullback of the following diagram in Cat

$$\begin{tikzcd} \mathrm{Cat}_2 \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Cat}_2 \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Cat}_2 \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Cat}_2 \arrow[dl, "h_1"] \\ & \mathrm{Cat} & & \cdots & & \mathrm{Cat} & \end{tikzcd}, \tag{3.1}$$

with n appearances of Cat_2.

Definition 3.2. Let Lad_n denote the pullback of the diagram above.

Example 3.3. When $n = 2$ our pullback is simply

$$\begin{tikzcd} \mathrm{Lad}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_2 \arrow[d, "h_1"] \\ \mathrm{Cat}_2 \arrow[r, "\mathrm{Mor}"'] & \mathrm{Cat} \end{tikzcd}.$$

There is also a commutative diagram

$$\begin{tikzcd} \mathrm{Cat}_3 \arrow[r, "\mathrm{Mor}"] \arrow[d, "h_2"'] & \mathrm{Cat}_2 \arrow[d, "h_1"] \\ \mathrm{Cat}_2 \arrow[r, "\mathrm{Mor}"'] & \mathrm{Cat} \end{tikzcd}.$$

Thus there is an induced functor $Ł : \mathrm{Cat}_3 \to \mathrm{Lad}_2$:

$$\begin{tikzcd} \mathrm{Cat}_3 \arrow[dr, dashed, "\exists ! Ł"'] \arrow[drr, bend left, "\mathrm{Mor}"] \arrow[ddr, bend right, "h_2"'] & & \\ & \mathrm{Lad}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_2 \arrow[d, "h_1"] \\ & \mathrm{Cat}_2 \arrow[r, "\mathrm{Mor}"'] & \mathrm{Cat} \end{tikzcd}$$

The ladder of a strict 3-category $Ł(\mathcal{C})$ is given by the homotopy 2-categories $h_2^{(0)}(\mathcal{C})$ and $h_2^{(1)}(\mathcal{C})$.

Example 3.4. Generalizing the previous example, for each $k \le n-2$ there is a diagram

$$\begin{tikzcd} \mathrm{Cat}_{n+1} \arrow[r, "h_2^{(k+1)}"] \arrow[d, "h_2^{(k)}"'] & \mathrm{Cat}_2 \arrow[d, "h_1"] \\ \mathrm{Cat}_2 \arrow[r, "\mathrm{Mor}"'] & \mathrm{Cat} \end{tikzcd}. \tag{3.2}$$

This induces a functor $Ł : \mathrm{Cat}_{n+1} \to \mathrm{Lad}_n$ whose image at an n-category $\mathcal{C}$ is the sequence of 2-categories

$$h_2^{(0)}(\mathcal{C}), \quad h_2^{(1)}(\mathcal{C}), \quad \ldots, \quad h_2^{(n-1)}(\mathcal{C}), \quad h_2^{(n-2)}(\mathcal{C})$$

.

Proposition 3.5. *The functor* $Ł : \mathrm{Cat}_{n+1} \to \mathrm{Lad}_n$ *is faithful.*

Proof. The image of a functor $F : \mathcal{C} \to \mathcal{D}$ in Cat_{n+1} is the sequence of its 2-truncations $h_2^{(k)}(F)$, which act as F on k-, $(k+1)$-, and in the classes classes of $(k+2)$-morphisms. In particular, the action of $h_2^{(k)}(F)$ on $(k+1)$-morphisms is given by F on the nose.

Thus, for another functor G, the equality $h_2^{(k)}(F) = h_2^{(k)}(G)$ implies that their action on $(k+1)$-morphisms is equal. Additionally, if $h_2^{(0)}(F) = h_2^{(0)}(G)$ and $h_2^{(n-2)}(F) = h_2^{n-2}(G)$ then they also agree on objects and n-morphisms too. So if F and G agree on all 2-truncations they are equal. □

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *isofibration* if any isomorphism $Fc \cong d$ in $\mathcal{D}$ can lifts to an isomorphism $c \cong c'$ in $\mathcal{C}$.

Lemma 3.6. *The functor* Mor *is an isofibration.*

Proof. An isomorphism $\phi : \mathrm{Mor}(\mathcal{B}) \cong \mathcal{C}$ corresponds to a collection of isomorphisms $\phi_{xy} : \mathcal{B}(x,y) \cong \mathcal{C}_{xy}$. Define $\mathcal{B}'$ to be the $(n+1)$-category with the same objects as $\mathcal{B}$ and with hom-n-categories

$$\mathcal{B}'(x,y) := \mathcal{C}_{x,y},$$

and with identities and composition induced by the components of ϕ. Then it's clear that there is an isomorphism $\Phi : \mathcal{B} \cong \mathcal{B}'$ lifting ϕ. □

Let Lad_n denote the category of ladders of small 2-categories.

Proposition 3.7. *The category* Lad_n *is accessible.*

Proof. Since both Mor and h_1 are accessible, so is the bilimit of the diagram in Equation 3.1 is an accessible category [MP89]. However, since Mor is an isofibration, a strict pullback involving it is equivalent to the pseudopullback of the same diagram, and hence itself a bilimit [MP89]. Applying this to Lad_n shows that it is accessible. □

Corollary 3.8. *The functor* $Ł : \mathrm{Cat}_{n+1} \to \mathrm{Lad}_n$ *is accessible.*

Proof. The homotopy 2-category functors are accessible (combine Propositions 2.14 and 2.16), hence so is the functor Ł induced by the diagram in Equation 3.2. □

Remark 3.9. When considering the canonical model structure on Cat and Cat_2 [R96, L04] functors h_1 and $\mathrm{Mor} : \mathrm{Cat}_2 \to \mathrm{Cat}$ preserve isofibrations and weak equivalences. In particular, Mor is a right Quillen functor. On other other hand, h_0 isn't, as it's not an adjoint (see Remark 2.10). Even then, it is a homotopical functor since it sends biequivalences to equivalences. I am not sure about the best way to proceed in this case.

Remark 3.10. The constructions in this section could be extended to the category Bicat_s of bicategories and strict pseudofunctors. In fact, the accessibility of h_1 and Mor could be proven similarly, so the analogous category of ladders would be accessible.

Although it sounds restrictive to work with bicategories and strict 2-functors, this is equivalent to working with pseudofunctors at least in the level of homotopy [L04, Theorem 4.6]. As in the previous remark, the functors involved are all homotopic, and $\mathrm{Mor} : \mathrm{Bicat}_s \to \mathrm{Cat}$ is a right Quillen functor, but the author doesn't know how to proceed with this knowledge.

4. Digressions on dualizability

Recall that internally to any bicategory we can define adjunctions and equivalences [JY20, Section 6.1 and 6.2]. As explained in the Appendix B, to study these (separately) it suffices to consider gaunt 2-categories, so we can ignore coherence issues that would otherwise show up.

Definition 4.1. A bicategory $\mathcal{B}$ is

(1) **adjunctable** if the 1-morphisms of $\mathcal{B}_\ell$ have left and right adjoints.
(2) a **2-groupoid** if the 1-morphisms of $\mathcal{B}_\ell$ are equivalences for every $\ell \leq k$ and if all 2-morphisms are invertible.

Definition 4.2. A ladder of 2-categories $\mathcal{B}_\bullet$ is **k-adjunctable** (resp. **k-groupoidal**) if each $\mathcal{B}_\ell$ is adjunctable (resp. a 2-groupoid) for $\ell \leq k$. If k is the length of $\mathcal{B}_\bullet$, we say it is **fully adjunctable** (resp. **groupoidal**).

A similar definition can be made for n-categories. The main motivation for the study of ladder is the following observation, which is still true for weak n-categories due to the discussion in the Introduction:

Proposition 4.3. *An n-category $\mathcal{C}$ is fully adjunctable iff $\text{Ł}(\mathcal{C})$ is fully adjunctable.*

We now turn to three well known simple statements about weak n-categories, making it explicit that they actually concern the underlying ladder.

Lemma 4.4. *Let $\mathcal{B}_\bullet$ be a k-adjunctable ladder and suppose that $\mathcal{B}_k$ is a 2-groupoid. Then $\mathcal{B}_\bullet$ is k-groupoidal.*

Proof. If $k = 1$ the proposition is vacuous. If $k > 1$, it suffices to prove that $\mathcal{B}_{k-1}$ is a 2-groupoid, for then the result follows by induction.

Let $f : x \to y$ be a morphism in $\mathcal{B}_{k-1}$ with a right adjoint $g : y \to x$, and unit and counit maps

$$[u] : fg \to 1_y \quad \text{and} \quad [v] : 1_x \to gf.$$

By hypothesis u and v are equivalences as 1-morphisms in $\mathcal{B}_k$. This means that there exist 1-morphisms $u^{-1} : 1_y \to fg$ and $v^{-1} : gf \to 1_x$ and invertible 2-isomorphisms

$$uu^{-1} \cong 1_{fg}, \quad u^{-1}u \cong 1_{1_y}, \quad vv^{-1} \cong 1_{gf} \quad \text{and} \quad v^{-1}v \cong 1_{1_x}.$$

Upon taking isomorphism classes, this shows that $[u]$ and $[v]$ are invertible as 2-morphisms in $\mathcal{B}_k$, so the adjunction at hand is an equivalence. □

This lemma proves the following folkloric statement:

Corollary 4.5. *Let $\mathcal{C}$ be a k-dualizable (∞, n)-category, with $k > n$. Then $\mathcal{C}$ is an ∞-groudoid.*

Let Lad_ω denote the category of ladders with infinite length.

Lemma 4.6. *If a ladder $\mathcal{B}_\bullet$ in Lad_ω is fully dualizable, then it is a ladder of 2-groupoids.*

Proof. A morphism $f : x \to y$ in $\mathcal{B}_k$ is an equivalence if there is a morphism $g : y \to x$ and invertible 2-morphisms

$$[\varepsilon] : ff^{-1} \to 1_y \text{ and } [\eta] : 1_x \to f^{-1}f.$$

Invertibility of $[\varepsilon]$ means that it has an inverse $[\varepsilon^{-1}] : 1_y \to ff^{-1}$ with respect to horizontal composition in $\mathcal{B}_k$. The equation $[\varepsilon^{-1}\varepsilon] = \mathrm{id}_{1_{ff^{-1}}}$ translates to the existence of an isomorphism between them when seen as 1-morphisms in $\mathcal{B}_{k+1}$.

In other words, in $\mathcal{B}_{k+1}$ there exist 2-morphisms $[A] : \varepsilon\varepsilon^{-1} \leftrightarrows 1_{fg} : [A^{-1}]$ such that $[AA^{-1}] = [1_{\varepsilon\varepsilon^{-1}}]$ and $[A^{-1}A] = [1_{1_{ff^{-1}}}]$. Equivalently, ε is an equivalence in $\mathcal{B}_{k+1}$.

If $\mathcal{B}_\bullet$ was finite, this process would eventually truncate into an actual equality (e.g. if it was a 3-ladder we would have $A^{-1}A = 1_{1_{ff^{-1}}}$). But it isn't, so we never have any equation imposed on the higher morphisms witnessing that f is an equivalence. So full dualizability and grupoidality coincide. $\square$

With this we uncovers the main result of [C07]:

Corollary 4.7. *An ω-precategory with duals is an ω-pregroupoid.*

The following statement is an adaptation of [DSPS13, Lemma 1.4.4] to the context of ladders. See there for a proof.

Lemma 4.8. *Let $(\mathcal{B}_0, \mathcal{B}_1)$ be a ladder and suppose that a 1-morphism f in $\mathcal{B}_0$ has a right adjoint f^R in $\mathcal{B}_0$ whose unit and counit have left adjoints as morphisms in $\mathcal{B}_1$. Then their left adjoints exhibit f^R as a left adjoint of of f.*

This is the original statement:

Corollary 4.9. *Let $\mathcal{C}$ be a 3-category. Let $f : x \to y$ be a 1-morphism in $\mathcal{C}$, and suppose that f admits a right adjoint f^R, with unit and counit maps $u : \mathrm{id}_x \to f^R \circ f$ and $v : f \circ f^R \to \mathrm{id}_y$. If u and v admit left adjoints u^L and v^L, then the 2-morphisms v^L and u^L, as unit and counit maps respectively, exhibit f^R as a left adjoint to f.*

5. Ladders of gaunt 2-categories

For certain questions about dualizability in higher categories, it suffices to consider ladders of gaunt 2-categories (see the Appendix B), hence in this section we study ladders of gaunt 2-categories.

The k-th homotopy 2-category functor $h_2^{(k)} : \mathrm{Cat}_n \to \mathrm{Cat}_2$ restricted to Gaunt_n is isomorphic to the functor that just picks the k, $k+1$ and $k+2$ morphisms (up to iso-$(k+3)$-morphisms), since all isomorphism classes have a single element. A similar remark holds for h_1.

A *ladder of gaunt 2-categories* is a ladder of 2-categories whose entries are gaunt. We define their category $\mathrm{Lad}_n^{\mathcal{G}}$ as the pullback of the diagram

(5.1)

$$\begin{tikzcd}[column sep=small]
\mathrm{Gaunt}_2 \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Gaunt}_2 \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Gaunt}_2 \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Gaunt}_2 \arrow[dl, "h_1"] \\
& \mathrm{Gaunt} & & \cdots & & \mathrm{Gaunt} &
\end{tikzcd}.$$

For our purposes we can regard a ladder of gaunt 2-categories as a sequence $(\mathcal{B}_0, \mathcal{B}_1, \dots)$ of gaunt 2-categories such that the hom-categories of $\mathcal{B}_k$ is equal to the category of objects and morphisms of $\mathcal{B}_{k+1}$.

We will need the following lemma to show that $\mathrm{Lad}_n^{\mathcal{G}}$ is locally presentable:

Lemma 5.1. *Let $\mathcal{V}$ and $\mathcal{W}$ be monoidal closed categories and $F : \mathcal{V} \leftrightarrows \mathcal{W} : G$ be an adjunction of lax monoidal functors.*[6] *Then the induced pair $F_* : \mathcal{V}\,\mathrm{Cat} \leftrightarrows \mathcal{W}\,\mathrm{Cat} : G_*$ is an adjunction.*

Proof. The data of $\mathcal{W}$-functor $f : F_*(\mathcal{C}) \to \mathcal{D}$ consists of a function $f_0 : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ and a morphism $\hat{f} : F(\mathcal{C}(x, y)) \to \mathcal{D}(f(x), f(y))$ for each pair of objects x and y in $\mathcal{C}$. Transposing each $\hat{f}$ under the adjunction we obtain a morphism $\hat{f}^\flat : \mathcal{C}(x, y)) \to G(\mathcal{D}(f(x), f(y))$, which together with f_0 assembles into the transposed $\mathcal{V}$-functor $f^\flat : \mathcal{C} \to G_*(\mathcal{D})$. □

Lemma 5.2. *When restricted to gaunt n-categories the functors h_0, h_1 and $h_2^{(k)}$ are right adjoints.*

Proof. The functor $h_0 : \mathrm{Gaunt} \to \mathrm{Set}$ is a right adjoint to the discrete category functor.[7] Then the result for h_1 and $h_2^{(k)}$ follows from Lemmas 2.15 and 5.1. □

Let $\mathrm{Lad}_n^{\mathcal{G}}$ denote the category of ladders of small gaunt n-categories.

Proposition 5.3. *The category $\mathrm{Lad}_n^{\mathcal{G}}$ is locally presentable.*

Proof. The category of locally presentable categories and right adjoints is closed under bilimits [B84]. Thus $\mathrm{Lad}_n^{\mathcal{G}}$ is locally presentable as it is a bilimit of right adjoints and Mor is an isofibration (see the proof of Proposition 3.7). □

Again there is a functor $Ł^{\mathrm{gaunt}} : \mathrm{Gaunt}_{n+1} \to \mathrm{Lad}_n^{\mathcal{G}}$ taking a gaunt $(n+1)$-category to its ladder of gaunt 2-categories. Since all functors involved are right adjoints, so is $Ł^{\mathrm{gaunt}}$.

Question 5.4. *What is the left adjoint?*

Our motto is that full dualizability of a strict n-category can be answered in terms of the ladder

$$h_2^{(0)}(\mathcal{C}), \quad h_2^{(1)}(\mathcal{C}), \quad \dots, \quad h_2^{(n-1)}(\mathcal{C}), \quad h_2^{(n-2)}(\mathcal{C}).$$

The result at the end of the Appendix B shows that it suffices to consider the ladder of gaunt 2-categories

$$\mathrm{gaunt}(h_2^{(0)}(\mathcal{C})), \quad \mathrm{gaunt}(h_2^{(1)}(\mathcal{C})), \quad \dots, \quad \mathrm{gaunt}(h_2^{(n-1)}(\mathcal{C})), \quad \mathrm{gaunt}(h_2^{(n-2)}(\mathcal{C})).$$

We believe that this construction extends abstractly to a functor $\mathrm{Lad}_n \to \mathrm{Lad}_n^{\mathcal{G}}$, but at this point we haven't filled the details. An initial obstruction is that obvious cube that would induce this functor doesn't commute because Mor doesn't play well with gauntification. Instead, we consider another functor $\mathrm{Cat}_n \to \mathrm{Lad}_n^{\mathcal{G}}$:

[6] In this setup $F \dashv G$ is monoidal by doctrinal adjunction [K01].

[7] One is tempted to say that h_0 is equal to the set of objects functor $\mathrm{ob} : \mathrm{Cat} \to \mathrm{Set}$ when restricted to gaunt categories, but that is not quite true, even though there is a natural isomorphism between them.

Definition 5.5. The functor $Ł^{\mathcal{G}} : \mathrm{Cat}_n \to \mathrm{Lad}_n^{\mathcal{G}}$ is the composition of $\mathrm{gaunt} : \mathrm{Cat}_{n+1} \to \mathrm{Gaunt}_{n+1}$ with $Ł^{\mathcal{G}} : \mathrm{Gaunt}_{n+1} \to \mathrm{Lad}_n^{\mathcal{G}}$.

Proposition 5.6. *The functor $Ł^{\mathcal{G}}$ is accessible.*

Proof. It suffices to check that $h_2 : \mathrm{Gaunt}_n \to \mathrm{Gaunt}_2$ is accessible. Similarly to Proposition 2.14, there is a diagram

$$\begin{tikzcd}
\mathrm{Cat}_n \arrow[dr, shift left, "(-)/\sim"] \arrow[dd, "h_0"'] & \\
& \mathrm{Gaunt}_n \arrow[ul, hook, shift left] \arrow[ul, phantom, "\perp"] \arrow[dl, shift right, "h_0"'] \\
\mathrm{Set} \arrow[ur, shift right, "D"'] \arrow[ur, phantom, "\top"] &
\end{tikzcd}$$

exhibiting h_0 as a composite of a left and a right adjoint, so it is accessible. Since $h_2 : \mathrm{Gaunt}_n \to \mathrm{Gaunt}_2$ is doubly induced by enrichment by h_0, the result follows from Lemma 2.13. □

Proposition 5.7. *The functor $Ł^{\mathcal{G}} : \mathrm{Cat}_3 \to \mathrm{Lad}_2^{\mathcal{G}}$ is not essentially surjective.*

Proof. The ladder defined by the 2-categories

$$\mathcal{B}_1 = \begin{tikzcd}
x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & y \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "g'"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, bend left=60, "h"] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & z
\end{tikzcd}$$

and

$$\mathcal{B}_2 = \begin{tikzcd}
f \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow A"] & f'
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow B"] & g'
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[r, bend left] \arrow[r, bend right] & h'
\end{tikzcd}\ .$$

is not in the image of Ł. In fact, a gaunt 3-category with this ladder would have 3-morphisms $A : x \to y$ and $B : y \to z$, but with no 3-morphism $C : x \to z$, impossible because a horizontal composition must be defined (and this can't be salvaged by isomorphisms in Lad_2). □

6. Ladders of $(\infty, 2)$-categories

In this section we sketch ideas about ladders of complete 2-fold Segal spaces, which are models of $(\infty, 2)$-categories. Most of the work remains to be done.

Let Δ denote the category of finite ordinals [0], [1], [2], etc. and order preserving maps between them.[8] Recall that a *simplicial object* in a category $\mathcal{C}$ is a presheaf in Δ with coefficients in $\mathcal{C}$. We denote their category by $\mathcal{C}^{\Delta^{\mathrm{op}}}$.

[8]In [MP89] the category $[n]$ would be denoted by $\mathbb{n} + \mathbb{1}$.

If X is a simplicial object in a complete category, we can form the pullback of the diagram below, denoted by $X_1 \times_{X_0} \cdots \times_{X_0} X_1$:

$$\begin{tikzcd}[column sep=small]
X_1 \arrow[dr, "d"'] & & X_1 \arrow[dl, "d_1"] \arrow[dr, "d_0"'] & & X_1 \arrow[dl, "d_0"] \arrow[dr, "d_1"'] & & X_1 \arrow[dl, "d_0"] \\
& X_0 & & \cdots & & X_0 &
\end{tikzcd}$$

For $k < n$, consider the map $d_{k,k+1} : [1] \to [n]$ sending $(0,1)$ to $(k, k+1)$. Then the first commuting square below evidently commutes, implying that so does the second:

$$\begin{tikzcd}
{[n]} & {[1]} \arrow[l, "d_{k+1,k+2}"'] \\
{[1]} \arrow[u, "d_{k,k+1}"] & {[0]} \arrow[l, "d_0"] \arrow[u, "d_1"']
\end{tikzcd}
\implies
\begin{tikzcd}
X_n \arrow[r, "d^{k+1,k+2}"] \arrow[d, "d^{k,k+1}"'] & X_1 \arrow[d, "d^1"] \\
X_1 \arrow[r, "d^0"'] & X_0
\end{tikzcd}$$

The *Segal maps* are the induced morphisms $\gamma_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

There is a strong connection between category and simplicial homotopy theory bridged through the Segal maps. For instance, a simplicial set is the nerve of a category if and only if the Segal maps are morphisms.

A similar statement holds for double categories as simplicial objects in Cat. In this case, to obtain 2-categories we need to impose discreteness of $X_0 \in \mathrm{Cat}$, and these ideas lead to Tamsamani's weak 2-categories and Paoli-Pronk's weakly globular 2-fold categories.

Applying the homotopy hypothesis and replacing Set with sSet_Q (with its classical model structure), we obtain a model for $(\infty, 1)$-categories as simplicial objects in sSet_Q with discrete X_0 and such that the Segal maps are weak equivalences. There is also an inductive definition from Segal categories to Segal n-categories, which model (∞, n)-categories.

We will follow the similar yet different approach of *complete Segal spaces* [JFS17, Section 2]. These satisfy a homotopical Segal axiom and are defined inductively, but are fundamentally different in the sense that there is an univalence axiom which can't be replicated when starting with sets.

Definition 6.1. A **Segal space** is a Reedy fibrant simplicial space $\mathcal{C} \in \mathrm{sSet}^{\Delta^{\mathrm{op}}}$ satisfying the following condition:

(1) *Segal axiom:* the Segal maps $\mathcal{C}_n \to \mathcal{C}_1 \times^h_{\mathcal{C}_0} \cdots \times^h_{\mathcal{C}_0} \mathcal{C}_1$ are weak equivalences in sSet_Q.

Given a Segal space $\mathcal{C}$ and 0-simplices $x, y \in \mathcal{C}_0$, consider the homotopy pullback $\mathcal{C}(x, y) := \{x\} \overset{h}{\underset{\mathcal{C}_0}{\times}} \mathcal{C}_1 \overset{h}{\underset{\mathcal{C}_0}{\times}} \{y\}$. The elements of $\mathcal{C}(x, y)$ are thus points $f \in \mathcal{C}_1$ with source and target connected to x and y, respectively.

The homotopy category $h_1(\mathcal{C})$ has the elements of $\mathcal{C}_0$ as its objects and $\pi_0(\mathcal{C}(x, y))$ as the hom-sets. The composition is defined by the following zig-zag of weak equivalences, which can be turned gives the composition upon applying π_0:

$$\begin{aligned}(\{x\} \overset{h}{\underset{\mathcal{C}_0}{\times}} \mathcal{C}_1 \overset{h}{\underset{\mathcal{C}_0}{\times}} \{y\}) \times (\{y\} \overset{h}{\underset{\mathcal{C}_0}{\times}} \mathcal{C}_1 \overset{h}{\underset{\mathcal{C}_0}{\times}} \{z\}) &\longrightarrow \{x\} \overset{h}{\underset{\mathcal{C}_0}{\times}} \mathcal{C}_1 \overset{h}{\underset{\mathcal{C}_0}{\times}} \mathcal{C}_1 \overset{h}{\underset{\mathcal{C}_0}{\times}} \{y\} \\ &\xleftarrow{\sim} \{x\} \overset{h}{\underset{\mathcal{C}_0}{\times}} \mathcal{C}_2 \overset{h}{\underset{\mathcal{C}_0}{\times}} \{y\} \\ &\xrightarrow{d_1} \{x\} \overset{h}{\underset{\mathcal{C}_0}{\times}} \mathcal{C}_1 \overset{h}{\underset{\mathcal{C}_0}{\times}} \{z\}.\end{aligned}$$

Associativity and unitality are built in through the simplicial identities.

Let $\mathcal{C}_{\mathrm{inv}} \subseteq \mathcal{C}_1$ denote the subspace of morphisms that are invertible in $h_1(\mathcal{C})$.

Definition 6.2. A Segal space $\mathcal{C}$ is **complete** if it satisfies the following condition:

(2) *Completeness:* the degeneracy $s_0 : \mathcal{C}_0 \to \mathcal{C}_{\mathrm{inv}} \subseteq \mathcal{C}_1$ is a weak equivalence.

Example 6.3. A topological space X is a complete Segal space when seen as a constant simplicial space.

Example 6.4. The nerve of a category satisfies the Segal axiom when seen as a discrete simplicial space. However, it is only complete if the category is gaunt.

To induct to (∞, n)-categories, consider the category of n-fold simplicial spaces $\mathrm{sSet}^{(\Delta^{\mathrm{op}})^n}$.

Definition 6.5. A **complete n-fold Segal space** is an n-fold simplicial space $\mathcal{C}_{\bullet,\dots,\bullet}$ such that...

(1) *n-uple axiom:* for every $1 \leq i \leq n$, and $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \geq 0$,

$$\mathcal{C}_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$$

is a Segal space.

(2) *n-fold axiom:* for every $1 \leq i \leq n$, and $k_1, \dots, k_{i-1} \geq 0$, the simplicial space

$$\mathcal{C}_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet}$$

is essentially constant, i.e. the degeneracy maps

$$\mathcal{C}_{k_1,\dots,k_{i-1}\bullet,0,\dots,0} \to \mathcal{C}_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$$

are weak equivalences.

(3) *Completeness axiom:* for every $1 \leq i \leq n$, and $k_1, \dots, k_{i-1} \geq 0$, the Segal space

$$\mathcal{C}_{k_1,\dots,k_{i-1},\bullet,0,\dots,0}$$

is complete.[9]

Example 6.6. The nerve of a double category satisfies (1) and (2) (see [P19]). However, it is only complete if the double category is a gaunt 2-category.

The analogy is that a 2-uple Segal space is akin to a double category,and a 2-fold one resembles a bicategory. The completeness axiom is a gauntness condition. This intuition can be understood through the following example.

[9]The usual definition requires that $\mathcal{C}_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$ is complete for all $k_{i+1}, \dots, k_n$, but it is actually equivalent to the one we state [JFS17, Lemma 2.8].

Example 6.7 (2-fold Segal spaces)**.** Given a 2-fold Segal space $\mathcal{C}_{\bullet,\bullet} : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{sSet}$, we use the define some names:

- $\mathcal{C}_{00}$ is the *space of objects*.
- $\mathcal{C}_{01}$ and $\mathcal{C}_{10}$ are the *space of vertical and horizontal 1-morphisms*, respectively.
- $\mathcal{C}_{11}$ is the *space of 2-cells*.
- $\mathcal{C}_{0\bullet}$ and $\mathcal{C}_{\bullet 0}$ are the *vertical and horizontal categories*, respectively.

Some explanations are in order:

(1) The 2-uple axiom yields several equivalences like

$$\mathcal{C}_{12} \longrightarrow \mathcal{C}_{11} \overset{h}{\underset{\mathcal{C}_{10}}{\times}} \mathcal{C}_{11}, \qquad \mathcal{C}_{03} \longrightarrow \mathcal{C}_{01} \overset{h}{\underset{\mathcal{C}_{00}}{\times}} \mathcal{C}_{01} \overset{h}{\underset{\mathcal{C}_{00}}{\times}} \mathcal{C}_{01},$$

and so on. For instance, the first one says that the space of "horizontal 2-simplices" is isomorphic to "composable pairs of horizontal morphisms", and so on. Compare it to the simplicial definition of strict n-fold categories in [P19].

(2) The n-fold axiom asserts that the degeneracy (i.e. identity) map

$$\mathcal{C}_{00} \longrightarrow \mathcal{C}_{01}$$

is an equivalence. This is a globularity axiom on the space of morphisms; for double categories, it asserts that the vertical category is trivial (up to equivalence).

(3) The completeness axiom states that there is a equivalence between the space of objects of $\mathcal{C}$ and its "invertible morphisms". This is a gauntness condition.

We now construct functors between the categories complete Segal spaces, analogously to Mor and h_1.

Lemma 6.8. *Let $\mathcal{C}$ be a complete $(n+1)$-fold Segal space. Then, for $k \geq 0$, both $\mathcal{C}_{\bullet,\dots,\bullet,0}$ and $\mathcal{C}_{k,\bullet,\dots,\bullet}$ are complete n-fold Segal spaces.*

Proof. The n-uple and completeness axioms are automatically satisfied in both cases. The n-fold axiom is also fulfilled because of the position where we put either 0 at the end and $k \geq 0$ in the beginning of $\mathcal{C}$ (any other entry would fail). $\square$

Definition 6.9. Let $\mathcal{C}$ be a (complete) $(n+1)$-fold Segal space. Then its

(1) **Underlying Segal space** is the n-fold (complete) Segal space $H_1(\mathcal{C})$ defined by $\mathcal{C}_{\bullet,\dots,\bullet,0}$.
(2) **Segal space of morphisms** is the n-fold (complete) Segal space $\mathrm{Mor}(\mathcal{C})$ defined by $\mathcal{C}_{1,\bullet,\dots,\bullet}$.

Remark 6.10. If we just wanted to consider the $(n-k)$-Segal space of morphisms between k-morphisms f and g, we could instead take the fiber product

$$\{f\} \overset{h}{\underset{\mathcal{C}_{1,\dots,1,0,\bullet,\dots,\bullet}}{\times}} X_{1,\dots,1,\bullet,\dots,\bullet} \overset{h}{\underset{\mathcal{C}_{1,\dots,1,0,\bullet,\dots,\bullet}}{\times}} \{g\},$$

with $k-1$ 1's.

Proposition 6.11. *There are functor H_1 and $\mathrm{Mor} : \mathcal{CSS}_{n+1} \to \mathcal{CSS}_n$ taking an $(n+1)$-fold complete Segal space to its underlying Segal space and to its Segal space of morphisms, respectively.*

Proof. The functors H_1 and Mor are induced by precomposition with the functors $(\Delta^{\mathrm{op}})^n \to (\Delta^{\mathrm{op}})^{n+1}$ defined by

$$([k_1], \ldots, [k_n]) \mapsto ([k_1], \ldots, [k_n], [0]) \quad \text{and} \quad ([k_1], \ldots, [k_n]) \mapsto ([1], [k_1], \ldots, [k_n]),$$

respectively. □

Definition 6.12. A **ladder of complete 2-fold Segal spaces** is a sequence of complete 2-fold Segal spaces $(\mathcal{B}_1, \ldots, \mathcal{B}_k)$ such that $H_1(\mathcal{B}_{k+1}) \cong \mathrm{Mor}(\mathcal{B}_k)$ for $0 \leq k \leq n-1$.

In an appropriate ambient ∞-cosmos [RV22], we can define the quasi-category of ladders $\mathrm{Lad}_n^{\mathcal{CSS}}$ as the homotopy pullback of the following diagram:

(6.1)
$$\begin{tikzcd}[column sep=small]
\mathcal{CSS}_2 \arrow[dr, "\mathrm{Mor}"'] & & \mathcal{CSS}_2 \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathcal{CSS}_2 \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathcal{CSS}_2 \arrow[dl, "h_1"] \\
& \mathcal{CSS} & & \cdots & & \mathcal{CSS} &
\end{tikzcd} .$$

Given a complete n-fold Segal space $\mathcal{C}$, we can also define the "k-th underlying complete 2-fold Segal space" $H_2^{(k)}(\mathcal{C})$ analogously to the k-th homotopy categories of Section 2. These can be done directly by

$$H_2^{(k)}(\mathcal{C}) = \mathcal{C}_{1,\ldots,1,\bullet,\ldots,\bullet,0,0},$$

with $k-1$ 1's, or by noticing that

$$H_2^{(k)}(\mathcal{C}) = H_1^{k-2} \circ \mathrm{Mor}^{n-k}.$$

Either way it's clear that there is a functor $H_2^{(k)} : \mathcal{CSS}_n \to \mathcal{CSS}_2$.

Remark 6.13. With strict n-categories the second definition corresponds to the commutative diagram

$$\begin{tikzcd}
\mathrm{Cat}_n \arrow[dd, "h_2^{(k)}"'] \arrow[dr, "\mathrm{Mor}^{n-k}"] & \\
& \mathrm{Cat}_k \arrow[dl, "H_1^{k-2}"] \\
\mathrm{Cat}_{(k,2)} &
\end{tikzcd}$$

where $H_1 : \mathrm{Cat}_{m+1} \to \mathrm{Cat}_m$ discard the non-invertible $(m+1)$-morphisms.

There are commutative diagrams

(6.2)
$$\begin{tikzcd}
\mathcal{CSS}_{n+1} \arrow[r, "H_2^{(k+1)}"] \arrow[d, "H_2^{(k)}"'] & \mathcal{CSS}_2 \arrow[d, "H_1"] \\
\mathcal{CSS}_2 \arrow[r, "\mathrm{Mor}"'] & \mathcal{CSS}
\end{tikzcd},$$

inducing a ladder functor $\text{Ł}^{\mathcal{CSS}} : \mathcal{CSS}_{n+1} \to \mathrm{Lad}_n^{\mathcal{CSS}}$.

There should be a functor $\mathrm{Lad}_n^{\mathcal{CSS}} \to \mathrm{Lad}_n^{\mathrm{Bicat}}$ taking a ladder of complete 2-fold Segal spaces to the corresponding ladder of homotopy bicategories. This expectation is related to the diagram

$$\begin{tikzcd} \mathrm{Cat}_n \arrow[dd, "h_2^{(k)}"'] \arrow[dr, "F"] & \\ & \mathrm{Cat}_{(n-k,2)} \arrow[dl, "h_2"] \\ \mathrm{Cat}_2 & \end{tikzcd} ,$$

where $\mathrm{Cat}_{(n-k,2)}$ is the category of $(n-k,2)$-categories, and F takes the category of morphisms $n-k$ times and then discards the non-invertible ℓ-morphisms for $\ell > 2$.

We haven't checked the details yet, but the idea is that a complete $(n+1)$-fold Segal space should be fully dualizable iff its image in $\mathrm{Lad}_n^{\mathrm{Bicat}}$ (or $\mathrm{Lad}_n^{\mathcal{G}}$) is fully dualizable. In any case, we expect the theory of ladders in $\mathcal{CSS}_2$ to be as well behaved as the theory of ladders of gaunt 2-categories, since completeness is a gauntness axiom.

7. Towards a bicategory of ladders

In this section we attempt to extend the category of ladders to a bicategory, However, it will soon be clear that the diagram analogous to the one in Equation 3.1 can't be drawn, as the true 2-categorical nature of h_1 and Mor live in different bicategories: the first concerns pseudonatural transformations, while the latter, icons.

In fact, there is another obstruction beforehand: the (ordinary) category of bicategories and pseudofunctors is not complete! See [L04, Example 4.5].

We refer to [JY20] for details about bicategories.

Definition 7.1. The **homotopy category** of a locally small bicategory $\mathcal{B}$ is the locally small category $h_1(\mathcal{B})$ with the same objects of $\mathcal{B}$ and isomorphism classes of morphisms as morphisms. The composition and identities are defined by choosing representatives.

Proposition 7.2. *With this construction $h_1(\mathcal{B})$ is a category.*

Proof. The composition is well defined because if there is a 2-isomorphism $\alpha : f \Rightarrow f' : x \to y$ and a morphism $g : y \to z$, then whiskering gives a 2-isomorphism $g.\alpha : gf \Rightarrow gf'$.

The components of the associator and unitors are 2-isomorphisms

$$\alpha_{fgh} : h(gf) \to (hg)f \quad , \quad \ell_f^x : 1_x f \to f \quad \text{and} \quad r_f^x : f1_x \to f$$

that, upon taking isomorphism classes, witness associativity and unitality in $h_1(\mathcal{B})$. □

Let Bicat^{ps} denote the bicategory of small bicategories, pseudofunctors and pseudonatural transformations, and $\underline{\mathrm{Cat}}$ the canonical 2-category of small categories.

Proposition 7.3. *There is a strict 2-functor $h_1 : \mathrm{Bicat}^{ps}$ to$\underline{\mathrm{Cat}}$ taking a small bicategory to its homotopy category.*

Proof. Given a pseudofunctor $F : \mathcal{A} \to \mathcal{B}$, define $h_1(F)$ by $h_1(F)(x) = Fx$ on objects and $h_1(F)(f) = [F(f)]$ on morphisms. Then $h_1(F)$ is a functor, since the components of the pseudonaturality constraints of F are 2-isomorphisms

$$F(g)F(f) \to F(gf) \quad \text{and} \quad 1_{Fx} \to F(1_x)$$

that, upon taking isomorphism classes, witness that $h_1(F)$ preserves composition and identities.

A pseudonatural transformation $\eta : F \Rightarrow G$ comes with components $\eta_x : Fx \to Gx$ which define a natural transformation $h_1(\eta) : h_1(F) \Rightarrow h_1(G)$ Indeed η is also equipped with 2-isomorphisms

$$\begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"'] \arrow[ur, phantom, "\Nearrow"] & Gy \end{tikzcd}$$

that witness the naturality of $h_1(\eta)$.

The functoriality of h_1 follows from noticing that the components of pseudofunctors and pseudonatural transformations are obtained by componentwise composition. □

Definition 7.4. The **category of morphisms** of a locally small bicategory $\mathcal{B}$ is the locally small category $\mathrm{Mor}(\mathcal{B})$ given by the disjoint union of its hom-categories.

Recall that an *icon* between pseudofunctors that agree on objects is an oplax transformation $\gamma : F \Rightarrow G$ with component identity 1-cells. Let Bicat_{psic} be the category of bicategories, pseudofunctors and icons [JY20, Thm 4.6.13].

Proposition 7.5. *There is a strict 2-functor* $\mathrm{Mor} : \mathrm{Bicat}^{psic} \to \mathrm{Cat}$ *taking a small bicategory to its category of morphisms.*

Proof. Given a pseudofunctor $F : \mathcal{A} \to \mathcal{B}$, define $\mathrm{Mor}(F)$ as the disjoint union of the components $\mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$, which are functors by construction.

An icon $\gamma : F \Rightarrow G$ presumes that F and G agree on objects and consists of natural transformations

$$\begin{tikzcd} \mathcal{A}(x,y) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{B}(Fx, Fy) = \mathcal{B}(Gx, Gy) \end{tikzcd} .$$

Then Mor takes γ to the disjoint union of its components.

The functoriality of Mor follows from noticing that the components of pseudofunctors and icons are obtained by componentwise composition. □

We can now explain the remark opening this section. Ideally, we would want to write a diagram mimicking the one in Equation 3.1, i.e.

$$\begin{tikzcd}[column sep=small] \mathrm{Bicat} \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Bicat} \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Bicat} \arrow[dl, "h_1"] \arrow[dr, "\mathrm{Mor}"'] & & \mathrm{Bicat} \arrow[dl, "h_1"] \\ & \mathrm{Cat} & & \cdots & & \mathrm{Cat} & \end{tikzcd} \tag{7.1}$$

However, Propositions 7.3 and 7.5 show there is not trivial way of doing this, since the 2-categorical domains h_1 and Mor disagree.

8. Further and other directions

There are several questions that remain open. Some of them regard Lad_n and its variants: is it a model category? What else can be done? What are some applications of the theory? Explaining full dualizability is useful, but there must be something more.

In the next three subsections we expand three possible directions that this project might take at some point.

Beyond full dualizability. Ladders are useful to deal with duals and adjunctions. Are there other concepts that they subsume?

Adjunctability in a bicategory $\mathcal{B}$ can be understood as a lifting problem

$$\begin{tikzcd} {[1]} \arrow[r] \arrow[d, "\iota"'] & \mathcal{B} \arrow[d] \\ \mathrm{Adj} \arrow[r] \arrow[ur, dashed] & * \end{tikzcd},$$

where Adj is the walking adjunction and ι includes picks either the left or right walking adjoint.

This feels like a fibration in a model category; let's try to make this idea happen. Say a *adj-fibration* is a functor $F : \mathcal{B} \to \mathcal{A}$ with the right lifting property against both ι's. Then a bicategory is adjunctable iff it is fibrant with respect to adj-fibrations. The ι's would be generating sets of cofibrations.

In general, the fully adjunctability of an n-category $\mathcal{C}$ is answered by similar diagrams (below on the left). Our motto is that $\mathcal{C}$ is adj-fibrant iff its ladder is so:

$$\begin{tikzcd} \Sigma^k([1]) \arrow[r] \arrow[d, "i"'] & \mathcal{C} \arrow[d] \\ \Sigma^k(\mathrm{Adj}) \arrow[r] \arrow[ur, dashed] & * \end{tikzcd} \iff \begin{tikzcd} \Sigma^k([1]) \arrow[r] \arrow[d, "i"'] & \text{Ł}(\mathcal{C}) \arrow[d] \\ \Sigma^k(\mathrm{Adj}) \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}.$$

This is an interesting lifting problem *answered by ladders*.

Are there any others, in particular unexpected or non-trivial ones?

Do ladders form an algebraic theory? Strict (small!) n-categories are models of a multisorted algebraic theory. This is not hard to see: the algebraic theory T has a set of generators $X = \{x_1, \dots, x_n\}$, products, and pullbacks, and the morphisms are those resembling source, target, and composition maps.

On the other hand, strict n-categories are a far cry from Lawvere theories, since pullbacks are essential - they are definitely not product theories. So we don't get a useful description for them in terms of monads. Still, this raises a natural question: are ladders of strict 2-categories models of an algebraic theory? The only part which can't be immediately accessed by an algebraic theory is h_1.

Higher groups. Let E be a connected space with two non-trivial homotopy group A, B and C in degrees n and $n+1$, and Y its truncation in degree $n+1$. Note that Y is classified by a class in $H^{n+2}(W, B)$, where W is its n-truncation.

Suppose $n \geq 3$ for simplicity. For instance, E might be a symmetric 2-group, which can be delooped until we a 7-type with non-trivial homotopy on degrees 5, 6 and 7.

Consider the Postnikov tower of E:

$$\begin{tikzcd}
n+2 & X = K(C, n+2) \arrow[r] & E \arrow[d] & \\
n+1 & W = K(B, n+1) \arrow[r] & Y \arrow[r, "\lambda"] \arrow[d] & K(C, n+3) \\
n & K(A, n) \arrow[r, "\sim"] & Z \arrow[r, "\kappa"] \arrow[d] & K(B, n+2) \\
n-1 & & * \arrow[r] & K(A, n+1)
\end{tikzcd}$$

Note that E is the total space of a fibration $X \to E \to Y$ that is classified by $H^{n+3}(Y, C)$.

Definition 8.1. The **first order Postnikov invariant** of E is the class in $H^{n+3}(W, C)$ obtained by restricting λ to W:

$$\begin{tikzcd}
W \arrow[r] \arrow[rr, bend right, "\alpha"'] & Y \arrow[r, "\lambda"] & K(C, n+3)
\end{tikzcd}$$

We want to try to express λ in terms of α and another class β in $H^{n+3}(Z, C)$. To that end we will employ the Serre spectral sequence of the fibration $W \to Y \to Z$.

Note that W is n-connected, so by Hurewicz theorem $H_k(W) = 0$ for $0 \geq k \geq n+1$, and $H_{n+1}(W)$ is B. Similarly $H_k(Z) = 0$ for $0 \geq k \geq n-1$, and $H_n(Z)$ is A. If there is no torsion, then $H^\bullet(-, C)$ is $\hom_{\mathbb{Z}}(H_\bullet(-), C)$ by the Universal Coefficient Theorem. In particular, $H^{n+1}(W, C)$ is $\hom_{\mathbb{Z}}(A, C)$ and $H^n(Z, C)$ is $\hom_{\mathbb{Z}}(B, C)$.

With can mpw sketch the E_2 page of the fibration $W \to Y \to Z$, and some higher differentials:

$$
\begin{tikzcd}
\alpha \in & H^{n+3}(W,C) \arrow[ddddddrrrrrrrr, "d_{n+4}"] &&&&&&&& \\
n+2 & H^{n+2}(W,C) \arrow[dddddrrrrrrr, "d_{n+3}"] &&&&&&&& \\
n+1 & \hom(A,C) \arrow[ddddrrrrrr, "d_{n+2}"] &&&&&&&& \\
n & 0 & 0 & \cdots & 0 &&&&& \\
& \vdots &&& \vdots &&&&& \\
& 0 &&& 0 &&&&& \\
& \mathbb{Z} & 0 & \cdots & 0 & \hom(B,C) & H^{n+1}(Z,C) & H^{n+2}(Z,C) & H^{n+3}(Z,C) & H^{n+4}(Z,C) \\
&&&& n-1 & n & n+1 & n+2 &&
\end{tikzcd}
$$

In the simplest case, the class κ in $H^{n+2}(Z,B)$ is trivial. Then Y is the product $W \times Z$, and by Kunneth we know that

$$H^{n+3}(Y,C) = \bigotimes_{i=0}^{n+3} H^i(W,C) \otimes H^{n-i}(Z,C).$$

However most of these are zero, so the expression reduces to

$$H^{n+3}(Y,C) = H^{n+3}(W,C) \oplus H^{n+3}(Z,C).$$

Note that α is in the first component. We claim that it is, in fact, the first component of λ. Then it remains to find a class β in $H^{n+3}(Z,C)$ to finish the description of λ and, hence, of E.

Definition 8.2. If κ is trivial, then the **second order Postnikov invariant** of E is the class β in $H^{n+3}(Z,C)$ obtained by projecting in the second component.

We want to give a more general definition of second order Postnikov invariant that holds even when κ isn't trivial. The question then would be: do the first and second order Postnikov invariants classify λ?

The relation of this subsection with the main project is more vague, but it exists. For instance, an n-groupoid corresponds to an n-type, and its truncations $h_2^{(k)}$ are 2-types / bicategories, which fit precisely in this paradigm of few sequential homotopy groups that we describe in this section.

Appendix A. Canonical model structures

In this section we provide a proof of the canonical model strutures on categories [R96] and 2-categories [L04] based on [M15]. For generalities on model structures, see [DS95, H99] or [T22, Section 1].

We will show that the canonical model structures Cat and Cat_2 arise by enrichment from the trivial model structure on Set and from Cat, respectively. Namely, we check that they are model structures of the following form:

Definition A.1. Let $\mathcal{V}$ be a closed symmetric monoidal model structure. The **Dwyer-Kan model structure** associated to $\mathcal{V}$, if it exists, is the model structure on $\mathrm{Cat}(\mathcal{V})$ such that

(1) the weak equivalences are the $\mathcal{V}$-functors $F : \mathcal{C} \to \mathcal{D}$ that are
  - *local weak equivalences*, i.e. the components $F_{xy} : \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ are weak equivalences.
  - *homotopically essentially surjective*, i.e. the functor $\pi_0(F) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ is essentially surjective.[10]

(2) the trivial fibrations are the functors that are
  - *surjective on objects*.
  - *local trivial fibrations*.

The proof will consists of checking the conditions in main result from [M15]:

Theorem A.2. *Let $\mathcal{V}$ be a combinatorial closed symmetric monoidal model category satisfying the Schwede-Shipley monoid axiom:*

- *The relative cell complexes obtained from the tensor product of a trivial cofibrations with any objects are weak equivalences.*

Then $\mathrm{Cat}(\mathcal{V})$ *admits the Dwyer-Kan model structure. Moreover, this model structure is combinatorial.*

Recall that the canonical model structure for Cat has equivalences of categories as weak equivalences and isofibrations as fibrations. A trivial fibration is an equivalence surjective on objects ([R96, Proposition 2.1]).

Theorem A.3. *The trivial model structure on* Set *induces the canonical model structure on* Cat.

Proof. Regarding the model structure on Set, the connected components functor $\pi_0 : \mathrm{Set} \to \mathrm{Set}$ is the identity, and so π_0 of a functor $F : \mathcal{C} \to \mathcal{D}$ is F itself. Thus a homotopically essentially surjective functor is an essentially surjective functor. Furthermore, a functor is locally a weak equivalence if it is componentwise bijective, i.e. fully faithful. So the induced weak equivalence in Cat are the equivalence of categories.

The induced trivial fibrations in Theorem A.2 are trivially the equivalences of categories that are surjective on objects. $\square$

Remark A.4. If $\mathcal{V}$ is cofibrantly generated, then the model structure in Theorem A.2 is also cofibrantly generated. If $f : v \to w$ is a generating (trivial) cofibration in $\mathcal{V}$, then the associated generating (trivial) cofibration in $\mathrm{Cat}(\mathcal{V})$ is the $\mathcal{V}$-functor

$$\begin{tikzcd} 0 \arrow[r, "v"] \arrow[d, dashed] \arrow[dr, phantom, "\downarrow f"] & 1 \arrow[d, dashed] \\ 0 \arrow[r, "w"'] & 1 \end{tikzcd}.$$

Together with the inclusion $\varnothing \to 0$, these give a set of generating cofibrations. For other generating trivial cofibrations, see [M15, Section 4].

[10]See the introduction of [M15] for a definition of π_0.

Example A.5. The trivial fibrations in Set are bijections, and note that a function of sets is

- surjective if it has the RLP against $\varnothing \to \{0\}$.
- injective if it has the RLP against $\{0, 1\} \to \{0\}$.

So these two morphisms form a set of generating cofibrations for Set. Then, by the previous remark, a trivial fibration in Cat is a functor with the RLP against the functors below.

$$\begin{tikzcd} \varnothing \arrow[d] \\ 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[d, dashed] & 1 \arrow[d, dashed] \\ 0 \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[d, dashed] & 1 \arrow[d, dashed] \\ 0 \arrow[r] & 1 \end{tikzcd}$$

In fact, one can check that a functor with the RLP against these functors is surjective on objects, full, and faithful, respectively, as expected.

The walking isomorphism $\mathbb{I} = \begin{tikzcd} 0 \arrow[r, shift left] & 1 \arrow[l, shift left] \end{tikzcd}$ is a generating Set-interval ([M15, Definition 4.10]). The generating trivial cofibration it generates is the inclusion $\theta_{\mathbb{I}} : 0 \hookrightarrow \mathbb{I}$, and an isofibration is, by definition, a functor that has the RLP against $\theta_{\mathbb{I}}$.

We now check that the canonical model structure on Cat induces the Lack model structure on 2-categories [L04]. In order to do so we have to check that the conditions of Theorem A.2 are still satisfied by Cat.

Lemma A.6. *The canonical model structure on* Cat *is monoidal closed.*

Lemma A.7. *A cofibration in* Cat *is a functor injective on objects.*

Proof. For the implication consider the lifting problem below, where G is a trivial fibration, i.e. an equivalence surjective on objects.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"'] & \mathcal{C}' \arrow[d, "G \in Fib \cap \mathcal{W}"] \\ \mathcal{D} \arrow[r] \arrow[ur, dashed] & \mathcal{D}' \end{tikzcd}$$

If F is injective on objects, then there is no choice for the objects in its image. For the objects not in the image, there is a lift since G is surjective on objects. Then any choice of lift works by extending it to morphisms.

We leave the converse as an exercise. □

Lemma A.8. *The canonical model structure on* Cat *satisfies the monoid axiom.*

Proof. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a trivial cofibration, which by the previous lemma is an equivalence injective on objects. Then, for any category $\mathcal{E}$, the product $\mathrm{id}_{\mathcal{E}} \times F$ is (trivially) an equivalence injective on objects, and hence a trivial cofibration.

Since trivial cofibrations are closed under pushouts, a cell attachment on a category $\mathcal{A}$ is also trivial cofibration:

$$\begin{tikzcd} \coprod_{i\in I}\mathcal{E}_i\times\mathcal{C} \arrow[r] \arrow[d, hook, "\sqcup \mathrm{id}_{\mathcal{F}}\times F"', "\in W\cap Fib"] & \mathcal{A} \arrow[d, "\in W\cap Fib"] \\ \coprod_{i\in I}\mathcal{E}_i\times\mathcal{D} \arrow[r] & \mathcal{F}_0 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

It remains to check that this stability still holds for a transfinite composition of cell attachments. This follows because the generating cofibrations in Remark A.5 are compact, so a transfinite composition of weak equivalences is again a weak equivalence [PS18, Lemma 2.2.(iv)]. ☐

Corollary A.9. *The canonical model structure on* Cat *induces the Lack model structure on* Cat_2.

Appendix B. Dualizability and gaunt categories

In this section we give some exposition on gaunt n-categories and expand the proof of [JFS17, Lemma 7.9], which states that gaunt 2-categories are enough to deal with existence problems on adjunctions.

Definition B.1. A **gaunt category** is an ordinary category with no non-trivial isomorphisms.

Gaunt categories can be interpreted as the *univalent categories* in the universe of sets, meaning they are the categories where the notions of isomorphism and equality coincide.

While this clearly imposes tremendous conditions on categories, it is remarkable that these constraints vanish upon changing to the universe of spaces (i.e. moving from ordinary to $(\infty,1)$-categories). In fact, the completeness axiom on Segal spaces is an univalence axiom.

Definition B.2. The **gauntification** of a category $\mathcal{C}$ is the gaunt category $\mathrm{gaunt}(\mathcal{C})$ defined by first identifying isomorphic objects, then identifying automorphisms with identities.

Remark B.3. The gauntification of a category $\mathcal{C}$ is a generalized congruence [BBP99] yielding the canonical quotient functor $\mathcal{C}\to\mathrm{gaunt}(\mathcal{C})$.

Gauntification defines a functor $\mathrm{gaunt}:\mathrm{Cat}\to\mathrm{Gaunt}$, where Gaunt is the full subcategory of gaunt categories.

Proposition B.4. *The inclusion* $\iota:\mathrm{Gaunt}\to\mathrm{Cat}$ *is a reflective subcategory inclusion.*

Proof. It's clear that $\iota\circ\mathrm{gaunt}$ is naturally isomorphic to $\mathrm{id}_{\mathrm{Gaunt}}$, so it remains to show that gaunt is a left adjoint. To see this, note that if $\mathcal{G}$ is a gaunt category then a functor $F:\mathcal{C}\to\mathcal{G}$ has to send isomorphic object to the same object, and then isomorphisms to the identity of that object. So F corresponds uniquely to a functor $\mathrm{gaunt}(\mathcal{C})\to\mathcal{G}$. ☐

Proposition B.5. *The category* Gaunt *is locally presentable.*

Proof. Since Cat is generated by the compact gaunt category $\bullet\to\bullet$, it suffices to check that the inclusion $\iota:\mathrm{Gaunt}\to\mathrm{Cat}$ commutes with directed colimits. In other words, we have to check that directed colimits of gaunt categories are gaunt.

A directed colimit $\mathcal{C} = \operatorname{colim} F_i$ in Cat is calculated by a generalized congruence relation [BBP99]. This roughly means that $\mathcal{C}$ is the disjoint union of F_i modulo relations; this can't produce new isomorphisms. $\square$

This assertion could instead be obtained from the following proposition.

Proposition B.6. *The left Bousfield localization of the canonical model structure on* Cat *at the inclusion* $i_0 : * \to J$*, where* J *is the walking isomorphism, yields* Gaunt *as its homotopy category.*

Proof. The left Bousfield localization of Cat at any finite set of morphisms is guaranteed because it is a (finitely presentable) combinatorial model category [B10]. A category $\mathcal{C}$ is fibrant in the localized model structure iff it is local with respect to i_0, i.e. the pullback

$$\mathrm{Iso}(\mathcal{C}) \cong \mathrm{Func}(J, \mathcal{C}) \xrightarrow{i_0^*} \mathrm{Func}(*, \mathcal{C}) \cong ob(\mathcal{C}).$$

is bijective. So $\mathcal{C}$ is gaunt.

In the canonical model structure, the cylinder object of a category $\mathcal{C}$ can be given by the following factorization of the codiagonal (see the other appendix):

$$\mathcal{C} \sqcup \mathcal{C} \xrightarrow{\mathrm{Cof}} \mathcal{C} \times J \xrightarrow{\in \mathcal{W}} \mathcal{C}$$

Hence two functors are homotopic if and only if they are naturally isomorphic. The cylinders are preserved by the left Bousfield localization, as it only increases weak equivalences and doesn't change the cofibrations, so this is still true after localizing. Thus the new homotopy category has gaunt categories as objects and functors up to natural isomorphism between them; but isomorphic functors between gaunt categories are equal.

$\square$

We can extend this discussion to strict n-categories.

Definition B.7. A **gaunt n-category** is a strict n-category $\mathcal{C}$ with no non-trivial k-morphisms for $1 \leq k \leq n$.

Similarly to Proposition B.5, we can prove:

Proposition B.8. *The category* Gaunt_n *is locally presentable.*

The concept of gaunt n-categories was introduced in [BSP21], where they play a crucial role in proving the unicity of theories of (∞, n)-categories. We refer the reader to that article for further details on gaunt n-categories.

We now turn to the proof of [JFS17, Lemma 7.9]. For details about adjunctions in bicategories, we refer the reader to [JY20, Section 6.1].

The gauntification $\mathrm{gaunt}(\mathcal{C})$ of a strict n-category $\mathcal{C}$ can be defined by first identifying isomorphic $(n-1)$-morphisms, then forcing the remaining automorphisms to be identities, and proceeding inductively. This yields a strict functor $\mathcal{C} \to \mathrm{gaunt}(\mathcal{C})$. We can then extend gauntification to bicategories by first finding an equivalent strict 2-category [JY20, Section 8.4], then gauntifying that.

Proposition B.9. *A 1-morphism f in a bicategory $\mathcal{B}$ has a left (resp. right) adjoint if and only if its class $[f]$ has a right (resp. right) adjoint in gaunt$(\mathcal{B})$.*

Proof. Adjoints are preserved by pseudofunctors [JY20, Proposition 6.1.7], including the strictification $\mathcal{B} \to \mathrm{st}(\mathcal{B})$ and the quotient $\mathrm{st}(\mathcal{B}) \to \mathrm{gaunt}(\mathrm{st}(\mathcal{B}))$; this establishes the implication. For the converse, first note that a morphism in $\mathcal{B}$ is a right adjoint iff it is dualizable in the equivalent 2-category $\mathrm{st}(\mathcal{B})$, so we can assume that $\mathcal{B}$ is strict.

If a morphism $[f] : [x] \to [y]$ in $\mathrm{gaunt}(\mathcal{B})$ is an identity, then it was originally an equivalence in $\mathcal{B}$, which can always be promoted to an adjoint equivalence. So suppose $f : x \to y$ is a non-invertible morphism such that $[f]$ has a left adjoint $[f]^L : [y] \to [x]$.

A lift $\widetilde{f^L} : y' \to x'$ of f to $\mathcal{B}$ by definition has isomorphisms

$$y \cong y' \xrightarrow{f} x' \cong x.$$

Denote this composition by $f^L : y \to x$.

Now let $\widetilde{\mathrm{ev}_f} : k \Rightarrow \ell$ and $\widetilde{\mathrm{coev}_f} : i \Rightarrow j$ be any lifts of the evaluation $\mathrm{ev}_{[f]} : [f^L f] \Rightarrow [\mathrm{id}_x]$ and $\mathrm{coev}_{[f]} : [\mathrm{id}_y] \Rightarrow [f f^L]$ to $\mathcal{B}$. Again there are iso-2-morphisms

$$f^L f \cong k \overset{\widetilde{\mathrm{ev}_f}}{\Longrightarrow} \ell \cong \mathrm{id}_x .$$

Let this composition be $\mathrm{ev}_f : f^L f \Rightarrow \mathrm{id}_x$, and define $\widehat{\mathrm{coev}_f} : \mathrm{id}_y \Rightarrow f f^L$ analogously. The triangle identity in $\mathrm{gaunt}(\mathcal{B})$

$$(1_{[f]} * \mathrm{ev}_{[f]}) \circ (\mathrm{coev}_{[f]} * 1_{[f]}) = 1_{[f]}$$

implies that

$$\psi := (1_f * \mathrm{ev}_f) \circ (\widehat{\mathrm{coev}_f} * 1_f) : f \Rightarrow f$$

is an iso-2-morphism, which can turn it to an identity by modifying $\widehat{\mathrm{coev}_f}$ through whiskering with ψ^{-1}. The other triangle identity

$$(\mathrm{ev}_{[f]} * 1_{[f]}) \circ (1_{[f]} * \mathrm{coev}_{[f]}) = 1_{[f]^L}$$

yields a 2-isomorphism $\phi : f^L \Rightarrow f^L$. But ϕ^2 has the other triangle identity in the middle,

$$\phi^2 = \begin{tikzcd}
& x \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \mathrm{ev}"] & & y \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \mathrm{ev}"] & & y \\
y \arrow[ur, "f^L"] \arrow[rr, equal] & & y \arrow[ur, "f^L"] \arrow[rr, equal] \arrow[u, phantom, "\Downarrow \mathrm{coev}"] & & y \arrow[ur, "f^L"'] \arrow[u, phantom, "\Downarrow \mathrm{coev}"] &
\end{tikzcd},$$

so $\phi^2 = \phi$. Since ϕ was invertible, it must be an identity too. □

Acknowledgments. Thanks Bob, Dorette and Theo for joining the committee. Also thanks to Claudia Scheimbauer for some sharp and insightful comments about complete Segal spaces. This work is supported by the Nova Scotia Graduate Scholarship program.

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