MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
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Homework Sheet 7
Due in: Wednesday 14th March, 1:30 PM

Compulsory questions

1 Solve the following recurrence relations. i.e. find an explicit formula for a_n in terms of n, and prove that it works. [You do not need to prove that your formula works if the equation is a second order constant-coefficient homogeneous linear recurrence.]

(a) $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = -1$, $a_1 = 2$.

(b) $a_n = 2a_{n-1} + 3(n-2)$, $a_0 = 1$. [Hint: try subtracting a_n from 2^n.]

(c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 5$, $a_1 = 3$.

(d) $a_n = \frac{1}{1+a_{n-1}}$, $a_0 = 1$. [You may use F_n to denote the nth Fibonacci number in your formula.]

2 Let F be a function defined by $F(0) = 1$, and

$$F(n) = \begin{cases} F\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \\ F(n-1) + 2 & \text{if } n \text{ is odd.} \end{cases}$$

Prove that $F(n)$ is odd for all natural numbers n.

3 (a) Give a recursive description of the number of ways of covering a $2 \times n$ chessboard with 2×1 tiles.

(b) Deduce that the number of ways to tile a $2 \times n$ chessboard is the $(n+1)$th Fibonacci number.

4 Let A be the Ackermann function; find $A(3, 5)$. [Hint: start by finding recurrence relations for $A(1, n)$ and then $A(2, n)$ and solving them.]

5 Suppose $k = 2^n$. Let $a_1, a_2, \ldots, a_k$ be a set of k positive real numbers. Let M_n be the maximum value of $\frac{a_1 a_2 \cdots a_k}{(a_1+a_2+\cdots+a_k)^k}$, for any positive real numbers $a_1, \ldots, a_k$.

(a) Show that M_n satisfies the recurrence $M_{n+1} \leqslant M_n^2 M_1^{(2^{n-1})}$. [Hint: rewrite the fraction $\frac{a_1 a_2 \cdots a_k}{(a_1+\cdots+a^k)^k}$ as

$$\frac{a_1 a_2 \cdots a_{2^{n-1}}}{(a_1+\cdots+a_{2^{n-1}})^{2^{n-1}}} \times \frac{a_{2^{n-1}+1} \cdots a_k}{(a_{2^{n-1}+1}+\cdots+a_k)^{2^{n-1}}}$$
$$\times \frac{(a_1+\cdots+a_{2^{n-1}})^{2^{n-1}} (a_{2^{n-1}+1}+\cdots+a_k)^{2^{n-1}}}{(a_1+\cdots+a_k)^k}$$

]

(b) Find the values of M_n. [Hint: for the $n = 1$ case, note that $(a_1 - a_2)^2 > 0$.]

Bonus Question

6 In the game Go, 2 players take turns to place stones of their colour (black or white) on the points in a 19×19 grid. A stone is captured if all of the neighbouring points (horizontally or vertically, but not diagonally) are occupied by stones of the opposite colour. On the other hand, if one of the neighbouring points is occupied by a stone of the same colour, then the two stones are considered like a single entity, i.e. they are both captured if all the neighbours of either stone (except for the ones on which the stones themselves are located) are occupied by stones of the opposite colour. Similarly for larger blocks of stones.

Give a recursive definition of when a stone is captured (possibly as part of a larger group of captured stones).