MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008
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Homework Sheet 6
Hints & Model Solutions

Compulsory questions

1 *We have 10 points in a square of side 3cm. Show that two of the points are within* $\sqrt{2}$*cm of each other.*

Divide the square into 9 1cm squares. By the pigeon-hole principle, some two of the points must be in the same 1cm square. Two points in a 1cm square must be within $\sqrt{2}$cm of each other.

2 *(a) You are dealt 13 cards. You may choose 5 of them to make a poker hand. Assuming you make the best poker hand possible from the 13 cards, what is the worst hand you can end up with (and why)? [The type of hand is sufficient – you don't need to give the exact cards.]*

Consider the number of ranks covered between the 13 cards. In order not to include a straight, for each 5-card set of ranks, one of them must be missing. Therefore, one of the ranks 2, 3, 4, 5, 6 must be missing, as must one of the ranks 7, 8, 9, 10, J. Therefore, in order not to get a straight, there must be at most 11 ranks. This leaves two repeated ranks, which can either form a two-pair or a 3 of a kind. Therefore, the worst hand that you can get is a two-pair.

(b) What if you are dealt 17 cards? [You still have to create a 5-card hand.]

Now by the pigeon-hole principle, there must be some 5 cards of the same suit, so it must be possible to make a flush. This is the worst case – i.e. it is possible that the best hand available is a flush.

3 *(i) Find the generating function for* $\binom{n}{k}k^2$ *as a function of* k*. [Hint: You know the generating function for* $\binom{n}{k}$*.]*

Let $f(x) = \sum_{k=1}^{n}\binom{n}{k}k^2x^k$. We know that $\sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n$. We can differentiate and multiply by x to get $\sum_{k=1}^{n}\binom{n}{k}kx^k = nx(1+x)^{n-1}$. Doing this again, we get $\sum_{k=1}^{n}\binom{n}{k}k^2x^k = nx(1+x)^{n-1}+n(n-1)x^2(1+x)^{n-2} = nx(1+nx)(1+x)^{n-2}$.

(ii) Deduce that $\sum_{k=1}^{n}\binom{n}{k}k^2 = 2^{n-2}n(n+1)$*. [Recall that we also showed this in a different way on sheet 2.]*

This sum is just the generating function from part (ii) evaluated at $x = 1$. This is $n(1+n)(1+1)^{n-1}$ as required.

4 At a party, there are n people. Each of them knows some number of the other people. Show that there are some two people at the party who know the same number of other people. [You may assume that if A knows B then B also knows A.]

The number of people a given person knows is between 0 and $n-1$, so there are n possibilities. However, if one person knows $n-1$ of the other people, then every other person knows that person, so there cannot be anyone who knows 0 of the other people. Therefore, the number of other people each person knows takes only $n-1$ values, so by the pigeon-hole principle, it takes the same value at least twice.

5 I have a sequence of integers $a_1, a_2, \ldots, a_n$ with the property that for any $i \leqslant j$, $a_i + a_{i+1} + \ldots + a_j$ is not a multiple of 17. What is the largest possible value of n. Justify your answer.

The largest value of n is 16 – for $i = 0, 1, \ldots, n$, let $S_i = \sum_{j=0}^{i} a_j$. Now the sum $a_i + a_{i+1} + \ldots + a_j$ is just the difference $S_j - S_{i-1}$. This is divisible by 17 if and only if $S_i \equiv S_j \pmod{17}$. There are 17 possible values for S_i modulo 17. Therefore, if there are 18 values for i, some two of the S_i must be congruent modulo 17. There are $n+1$ values for i, therefore, if $n + 1 \geqslant 18$, then we will get a multiple of 17. It is possible to get a set of numbers with $n = 16$ – take $a_1 = a_2 = \ldots = a_{16} = 1$.

6 There are 100 students in a class. There are 9 homework sheets.

(a) Suppose that for each homework sheet at least n students hand in solutions. How large does n have to be to ensure that there is some student who attempts all the homework sheets?

n must be at least 89 –

One way to see this is that if n is at least 89, then at least $89 \times 9 = 801$ solutions are handed in. 801 is more than 8×100; this means that there must be some student who hands in more than 8 solutions. Since there are only 9 assignments, this student must attempt all the assignments.

Alternatively, consider the set A_i of students who do not hand in the ith homework sheet. A student who attempts all homework sheets is not in any of the A_i, so we want the union of the A_i to not be the whole set. This will be ensured if each A_i has size at most 11, since there are 9 A_i, so their total size will be at most 99.

(b) Suppose now that each student submits solutions to at least m of the homework sheets. How large must m be to ensure that there are 2 students who submit solutions to the same set of homework sheets?

The number of possible choices for sets of homework sheets to attempt are: $\binom{9}{m} + \binom{9}{m+1} + \cdots + \binom{9}{9}$. If this is less than 100, then by the pigeon-hole principle, some two students must attempt the same set of homework sheets. Recall: $\binom{9}{9} = 1$, $\binom{9}{8} = 9$, $\binom{9}{7} = 36$, $\binom{9}{6} = 84$, so we get that $m = 7$.