MATH 2113/CSCI 2113, Discrete Structures II
Winter 2008

Toby Kenney

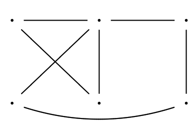
Homework Sheet 7

Due: Wednesday 19th March: 1:30 PM

Compulsory questions

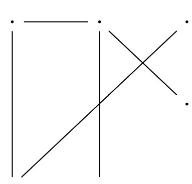
1 *Which of the following graphs have Euler circuits?*

(a)



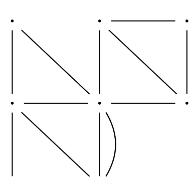
All vertices have even degree, so the graph has an Euler circuit.

(b)



There are vertices of odd degree, so there is no Euler circuit.

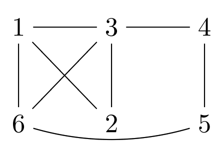
(c)



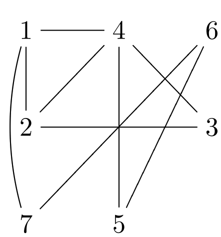
There are vertices of odd degree, so there is no Euler circuit.

2 *Find Hamiltonian circuits in the following graphs.*

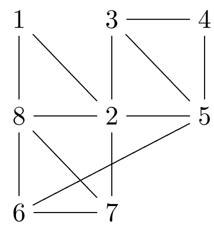
(a)



(b)

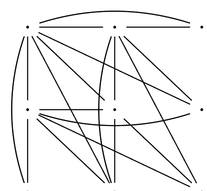


(c)



3 *(a) Draw a simple graph with vertices of degrees 2,2,3,3,4,5,6,6,7.*

One example is:

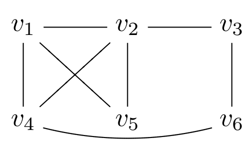


(b) Is there a simple graph with vertices of degrees 2,3,4,5,6,6,6,6,7?

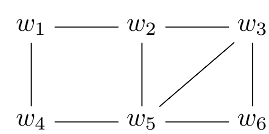
No. The total $2+3+4+5+6+6+6+6+7 = 45$ is odd, so there cannot be such a graph.

4 *Which of the following pairs of graphs are isomorphic? For pairs which are isomorphic, write down the isomorphism. For pairs which are not isomorphic, give a reason why they are not isomorphic.*

(a)



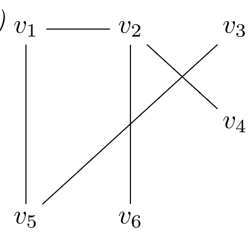
and



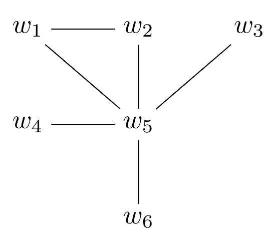
These are isomorphic with the following isomorphism:

v_1	w_3
v_2	w_5
v_3	w_4
v_4	w_2
v_5	w_6
v_6	w_1

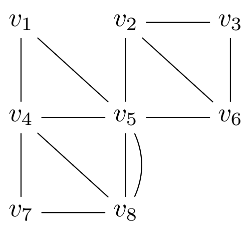
(b)

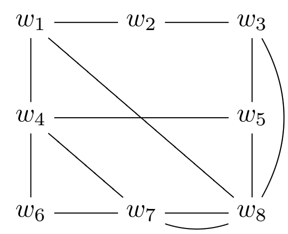


and



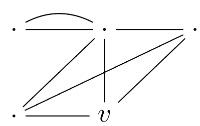
These are not isomorphic: the first graph has 5 edges, the second graph has 6.

(c) 

and 

These are not isomorphic. They both have 13 edges, but the first graph has 3 vertices of degree 2 (v_1, v_3 and v_7) while the second graph has only 2 (w_2 and w_6).

5 *Write down the adjacency matrix for the following graph. How many cycles of length 3 are there that pass through the vertex v?*



The adjacency matrix is

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

We get

$$A^2 = \begin{pmatrix} 4 & 0 & 2 & 2 & 2 \\ 0 & 7 & 2 & 2 & 2 \\ 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 14 & 4 & 4 & 4 \\ 14 & 6 & 11 & 11 & 11 \\ 4 & 11 & 6 & 7 & 7 \\ 4 & 11 & 7 & 6 & 7 \\ 4 & 11 & 7 & 7 & 6 \end{pmatrix}$$

v is represented by the bottom row and rightmost column. The number of walks of length 3 starting and ending at v is therefore 6. However, since there are no loops, a walk of length 3 that starts and ends at v cannot have a repeated vertex (since the two occurences of the same vertex have to be at least two places apart in the walk) so it must be a cycle of length 3. There are therefore 6 cycles of length 3 beginning and ending at v.

6 *For which values of m and n does the complete bipartite graph on m and n vertices have a Hamiltonian cycle? Justify your answer.*

Let the two sets of vertices in the bipartite graph be A and B. Since all edges go from a vertex in A to a vertex in B, the vertices in any cycle must alternate between the sets A and B. A Hamiltonian cycle contains each vertex exactly once, so since the vertices alternate between A and B, there must be the same number of vertices in A and B. (If the cycle starts in A, then the last vertex it visits must be in B, since it will then return to the first vertex, which is in A. Therefore, there must be exactly the same number of vertices in A and B – if we just wanted a Hamiltonian path, we could have one more vertex in A by starting and ending at different points in A.) Therefore, if there is a Hamiltonian cycle, then $m = n$.

On the other hand, if $m = n$, we can label the vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$, and there is a Hamiltonian cycle $a_1 b_1 a_2 b_2 \ldots a_n b_n a_1$, unless $n = 1$, when there is clearly no Hamiltonian cycle. Therefore, there is a Hamiltonian cycle if and only if $m = n \neq 1$.