MATH 2600/STAT 2600, Theory of Interest
FALL 2010
Toby Kenney
Homework Sheet 2
Due: Thursday 7th October: 1:00 PM

1. *Calculate the accumulated value on maturity of \$1,000 invested for 23 years at 5% interest compounded:*

   (a) annually.

$$1000(1.05)^{23} = 3071.52$$

   (b) monthly.

$$1000(1 + \frac{0.05}{12})^{23\times 12} = 3150.66$$

2. *Which of the following rates of interest is best for the investor:*

   (i) 10% compounded monthly

   (ii) 10.5% compounded annually

   (iii) 9.8% compounded daily

   We calculate the annual effective rate in each case:

   (i) $(1 + \frac{0.1}{12})^{12} - 1 = 0.1047 = 10.47\%$

   (ii) 10.5%

   (iii) $(1 + \frac{0.098}{365})^{365} - 1 = 0.1029 = 10.29\%$

   So (ii) is best for the investor.

   [and (iii) would best for the borrower.]

3. *A man wants to buy a car which costs \$20,000 today (including all taxes), and whose price is linked to inflation of 6%. He currently has \$17,000, which he plans to invest until he has enough to buy the car. Assume that the exact method is used to calculate interest for partial years. How long does he have to wait before he can afford to buy the car:*

   (a) if he can invest his money at 8% compounded annually?

   The real rate of interest is given by

$$\frac{8-6}{1.06} = 1.88679\ldots\%$$

   We want to know how long it takes the man's money to grow by a factor of $\frac{20000}{17000}$ at this rate of interest, compounded annually. We compute this as

$$\frac{\log(\frac{20000}{17000})}{\log(1.0188679\ldots)} = 8.6945$$

So 8.6945 years.

(b) if he can invest his money at 10% compounded annually?

The real rate of interest is given by

$$\frac{10-6}{1.06} = 3.77\ldots\%$$

We want to know how long it takes the man's money to grow by a factor of $\frac{20000}{17000}$ at this rate of interest, compounded annually. We compute this as

$$\frac{\log(\frac{20000}{17000})}{\log(1.0377\ldots)} = 4.3875$$

So 4.3875 years.

4. *An investor has a promissory note for \$10,000 in 8 years at 4% interest compounded monthly. After 3 years, 7 months, she sells it to a bank which discounts notes at 3% interest, also compounded monthly.*

   (a) How much does the bank pay for the note?

   The maturity value of the note is $10000(1+\frac{0.04}{12})^{12\times 8} = 13763.95$. Therefore, the bank discounts it over 4 years 5 months to get $13763.95(1+\frac{0.03}{12})^{-(12\times 4+5)} = 12057.88$.

   (b) What annual effective rate of return does the investor achieve?

   The investment increases by a factor of 1.205788 in $3+\frac{7}{12} = \frac{43}{12}$ years, so in every year, it increases by a factor of $1.205788^{\frac{12}{43}} = 1.0536$, which is a rate of return of 5.36%.

   (c) If the investor is looking to make a return of 5.5% annual effective rate on her investment, by what time does she need to sell it to the bank? [Use the exact method for fractional months.]

   After n months, the price the bank will pay for the note is $13763.95(1.0025)^{n-12\times 8}$, while the price the investor needs to sell it at to achieve a 5.5% annual effective rate is $10000(1.055)^{\frac{n}{12}}$. We need to find the value of n for which these prices are equal. That is:

$$
\begin{aligned}
10000(1.055)^{\frac{n}{12}} &= 13763.95(1.0025)^{n-96} && (1)\\
\log(10000) + \frac{n}{12}\log(1.055) &= \log(13763.95) + (n-96)\log(1.0025) && (2)\\
\log(10000) + 96\log(1.0025) - \log(13763.95) &= n\left(\log(1.0025) - \frac{\log(1.055)}{12}\right) && (3)\\
n &= \frac{\log(10000) + 96\log(1.0025) - \log(13763.95)}{\log(1.0025) - \frac{\log(1.055)}{12}} && (4)\\
&= 40.60 && (5)
\end{aligned}
$$

So she should sell it within 40.60 months, or 3 years 4.60 months.

5. *Mr. Smith invests \$10,000 at 5% interest, compounded annually. He plans to use the investment to pay a bill of \$4,000 after 7 years. What is the largest amount x that he can take out after 2 years 8 months, and still have enough to pay the \$4,000 after 7 years?*

   (a) Calculate x assuming the investment uses:

   (i) the exact method for partial time intervals.

   Using an equation of value with focal date at 2 years 8 months, we have that the original investment is worth $10000(1.05)^{\frac{8}{3}} = 11389.50$, while the \$4,000 at the end of 7 years has a value of $4000(1.05)^{-\frac{13}{3}} = 3237.723$, so he has to leave at least \$3,237.73 in the account. This means that he can take out \$8,151.77 after 2 years 8 months.

   (ii) the approximate method.

   Using the approximate method, the amount of money in the account after 2 years is $10000(1.05)^2 = 11025$, the amount after 2 years 8 months is $11025(1+0.05\frac{8}{12}) = 11392.50$. The discounted value of \$4,000 at this time, is given by discounting by 5 years, to get $4000(1.05)^{-5} = 3134.104\ldots$, then accumulating forwards using simple interest to get $3134.104\ldots(1 + 0.05\frac{8}{12}) = 3238.5748\ldots$. Since Mr. Smith wants to have at least \$4,000 in the account after the 7 years, he needs to leave at least \$3238.58 in the account after 2 years 8 months. This means he can take out \$8,153.93 after 2 years 8 months.

   (b) For the approximate method, does it matter which date is chosen as the focal date for the equation of value? If so, what is the correct choice?

   It is important to be careful, because simple interest is used for some parts of the calculation, and compound interest is used for other parts. Because simple interest is used for part of the calculations, it does matter which date is chosen as the focal date. Since we use compound interest to establish equivalent values to the \$10,000 and \$4,000 after 2 years in both cases, and then use simple interest to compare these to the value after 2 years 8 months, the values to which simple interest applies are only on two dates — after 2 years, and after 2 years 8 months. Therefore, either of these dates can be chosen as the focal date.

6. *A man takes out a 10 year loan for \$20,000 at 7% interest, compounded annually. After 4 years, the interest rate increases to 9%. He repays \$12,000 after 5 years, then after 7 years from the start of the loan, he repays a further \$4,000. Calculate the outstanding balance at the end of the 10 years.*

   After 4 years, the balance of the loan is $20000(1.07)^4 = 26215.92$. After this, the loan is at 9%, so after 5 years the balance is $26215.92 \times 1.09 - 12000 = 16575.35$. After 7 years, the balance is $16575.35(1.09)^2 - 4000 = 15693.18$. Now at the end of 10 years the balance is $17446.80(1.09)^3 = 17446.80$.

7. *A woman invests \$30,000 at 5% interest compounded annually. She wants to use this money to give \$20,000 to each of her children when they turn 18. She currently has 2 children aged 9 and 4, and she plans to have a third child. How long does she have to wait before having the third child, in order for the investment to be sufficient?*

   Using an equation of value at the current time, we see that her gifts to her first 2 children of 20000 after 9 years and 20000 after 14 years, have present value $20000(1.04)^{-9} + 20000(1.04)^{-14} = 22993.537\ldots$, so this leaves \$7,006.4626... of the original \$30,000 for the third child.

   We need to know how long it will take the leftover \$7,006.4626... to grow to \$20,000. We compute this as $\frac{\log(\frac{20,000}{7006.4626\ldots})}{\log(1.05)} = 21.50$ years. Since this is the time when the third child turns 18, she should wait about 3 and a half years before her third child is born.