MATH 2600/STAT 2600, Theory of Interest
FALL 2010
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Homework Sheet 3
Due: Thursday 14th October: 1:00 PM

1. *Ms. Jones invests \$500 a month in her pension plan, which pays interest at $j_{12} = 5\%$. She makes her first investment on her 21st bithday. How much money does she have in the plan when she retires on her 65th birthday (assuming that she still makes the payment on that birthday)?*

$$500\frac{(1+\frac{0.05}{12})^{44\times 12+1}-1}{\frac{0.05}{12}} = 962554.79$$

2. *Jack wants to start a pension plan for his retirement. He decides that after he retires, he will need \$1,000 a month from his pension. His pension fund pays interest at $j_{12} = 6\%$.*

   (a) If he plans to have enough in his pension fund to last for 25 years, how much will he need at the time he retires (assuming the payments start one month after his retirement).

$$1000\left(\frac{1-1.005^{-300}}{0.005}\right) = 155206.864$$

   So he will need \$155,206.87

   (b) He is currently 43. If he plans to retire at 65, i.e. exactly 22 years from today, how much does he need to contribute to the pension plan every month to acheive the target in part (a), assuming he makes his first payment today, and his last one in 22 years.

$$X\left(\frac{1.005^{12\times 22+1}-1}{0.005}\right) = 155206.87$$

$$X = \left(\frac{155206.87\times 0.005}{1.005^{12\times 22+1}-1}\right) = 282.216$$

   So he needs to contribute \$282.22 every month.

   (c) He decides that he can only afford to put in \$200 a month. He wants his pension fund to last until he is 90 years old — i.e. exactly 47 years from today, and he still wants to be able to withdraw \$1000 a month when he starts to withdraw from his pension. When can he retire and still achieve this?

   If he retires after n months, the value of the fund will be

$$200\left(\frac{1.005^n - 1}{0.005}\right)$$

He wants to be able to withdraw \$1,000 a month from this fund until age 90. To do this, he needs the value of the fund to be

$$1000\left(\frac{1 - 1.005^{n-(47\times 12+1)}}{0.005}\right)$$

We need to find the n for which these are equal, i.e.

$$200\left(\frac{1.005^n - 1}{0.005}\right) = 1000\left(\frac{1 - 1.005^{n-(47\times 12+1)}}{0.005}\right)$$

$$1.005^n - 1 = 5(1 - 1.005^n \times 1.005^{-(47\times 12+1)})$$

$$(1 + 1.005^{-(47\times 12+1)}))1.005^n = 6$$

$$1.005^n = \frac{6}{(1 + 1.005^{-(47\times 12+1)}))}$$

$$n = \frac{\log 6 - \log(1 + 1.005^{-(47\times 12+1)})}{\log 1.005} = 347.616$$

So he would be able to retire after just over 347 and a half months, i.e just before 28 years. (It makes sense for this n not to be an integer, as we can think of him working part of the month to earn money, and not having to withdraw so much the first month.)

3. *A man takes out a loan for \$10,000 at an interest rate* $j_4 = 10\%$. *He agrees to repay it with a set of quarterly payments of \$500.*

   (a) How many payments does he need to make?

$$500\frac{1 - 1.025^{-n}}{0.025} = 10000$$

$$1 - 1.025^{-n} = \frac{1}{2}$$

$$1.025^{-n} = \frac{1}{2}$$

$$n = \frac{\log 2}{\frac{\log}{1}.025} = 28.07$$

So he needs to make either 28 payments, where the last payment is increased, or 29 payments where the last payment is smaller.

(b) If he agrees to pay off the debt by making a balloon payment on the last payment, calculate the value of the last payment?

The first 28 payments have a discounted value of $500\frac{1-1.025^{-28}}{0.025} = 9982.444....$ therefore, the extra amount on the last payment should pay off the remaining 17.55567... of the loan. By the time he makes the 28th payment, this remainder is equivalent to a value of $17.55567 \times 1.025^{28} = 35.05$

So the final payment is \$535.05.

(c) If instead, he agrees to make a drop payment, what will be the value of his last payment?

If he makes a drop payment instead, he is still paying off the remaining 17.55567... of the original loan, but now it's value has increased to \$35.93, so the final payment is \$35.93.

4. *A woman takes out a mortgage for \$100,000. The bank tells her that she can pay it off with monthly payments of \$1,000 for 20 years. What nominal rate of interest (compounded monthly) are they charging her?*

We have:

$$100000 = 1000\frac{1-(1+i)^{-240}}{i}$$

or $100 = \frac{1-(1+i)^{-240}}{i}$. If we try a couple of values, we get the following:

$i = 0.01 \quad \frac{1-1.01^{-240}}{0.01} = 90.819...$
$i = 0.005 \quad \frac{1-1.005^{-240}}{0.005} = 139.58...$
$i = 0.007 \quad \frac{1-1.007^{-240}}{0.007} = 116.076...$
$i = 0.008 \quad \frac{1-1.008^{-240}}{0.008} = 106.53...$
$i = 0.009 \quad \frac{1-1.009^{-240}}{0.009} = 98.17...$

We can interpolate between these values to get our next guess $i = 0.008 + 0.001 \times \frac{106.53...-100}{106.53...-99.17...} = 0.00878...$, which is equivalent to $j_{12} = 10.54\%$.

If we repeat the interpolation to get a more accurate result, we find the rate of interest is 10.52%

(b) [bonus] Suppose that she can invest her money at $j_{12} = 7\%$. Suppose further, that the house price rises with inflation of of 0.5% every month. How long would she have to save up the same \$1,000 a month payments before she could buy the house with a single payment?

My first instinct was that this question could be done using the real rate of interest. However, because the payments do not increase with inflation, this doesn't work. The accumulated value of her savings after n months is

$$1000\frac{\left(1+\frac{0.07}{12}\right)^n - 1}{\frac{0.07}{12}}$$

while the price of the house is 100000×1.005^n. We need to find the n for which these are equal, or equivalently

$$\left(1+\frac{0.07}{12}\right)^n - 1 = \frac{7}{12} \times 1.005^n$$

$$\left(1+\frac{0.07}{12}\right)^n - \frac{7}{12} \times 1.005^n = 1$$

We can't solve this equation analytically, we can use trial and error, or linear interpolation, to get the first whole number larger than n, namely 128 months. This is how long she would have to save up.

(c) Suppose she were given the option of renting the same house while saving up the money to buy it. What monthly rent should she be willing to pay?

We are looking for the amount of rent such that if she were to save up the remainder of her \$1,000 monthly payments each month, by the end of 20 years, she would have enough money to buy the house. After 20 years, the house price would be $100000(1.005)^{240}$. To save up this much, she would need to put aside

$\frac{100000(1.005)^{240}\frac{0.07}{12}}{(1+\frac{0.07}{12})^{240}-1} = 635.445...$

each month. Thus, she could afford to pay monthly rent of \$365.555, if it were payable at the end of the month. Payable at the beginning of the month, it would be $365.56(1+\frac{0.07}{12})^{-1} = 362.44$

5. *Mr. MacDonald makes a donation of \$1,000,000 to his old University. He requests that it be used to set up a scholarship fund which pays \$20,000 a year to each of 3 students, and should last for 100 years. If the money is invested at 5% interest per year, how long does the university have to wait before it can make the first payment from the scholarship fund.*

   The discounted value of the scholarship fund, which is paying out \$60,000 a year, at 5% interest, for 100 years, is

$$60000\frac{1-1.05^{-100}}{0.05} = 1190874.61$$

   So we need to work out when the amount of money in the fund has reached this value, which is after $\frac{\log 1.19087461}{\log 1.05} = 3.58$ years. The above sum of money is the amount that needs to be in the account 1 year before the first payment, so the first payment can be made after 4.58 years.

6. *Dr. Proctor sets up a savings account on 1st January 2001 to pay for his childrens' education. He pays \$100 into the account each month, starting on 1st January 2001, until 1st June 2004, when he increases the payments to \$150 a month. Then from 1st April 2009 onwards, he reduces the*

payments back to $100 each month. The interest rate for the account is determined each year by market conditions, and the interest rates are as given in the following table:

Year	*Interest rate*
2001	*5%*
2002	*5%*
2003	*4%*
2004	*4%*
2005	*5.5%*
2006	*5.5%*
2007	*8%*
2008	*4%*
2009	*4%*
2010	*5%*
2011	*5%*
2012	*5%*

How much is in the account when he makes his last payment on 1st August 2012?

We divide the term of the investment into n periods as in the following table. The total can be calculated either by calculating a cumulative total after each set of payments (6th column), or by calculating the accumulated value of each set of payments at the end of the investment (7th column) and finding the total.

Dates	No. of payments	Monthly payment	Interest Rate	Value of these payments	Total value after these payments	Value of these payments at end of investment
01/12/00–	25	100	5%	$2,629.09	$2,629.09	$4240.799
01/01/03–	17	100	4%	$1746.10	$4,528.21	$2661.597
01/05/04–	7	150	4%	$1060.5585	$5,5695.48	$1579.3986
01/01/05–	24	150	5.5%	$3796.284	$10,152.43	$5065.85275
01/01/07–	12	150	8%	$1867.4889	$12,862.57	$2301.0372
01/01/08–	15	150	4%	$2303.26597	$15,824.19	$2699.797
01/03/09–	9	100	4%	$912.0938	$17,217.38	$1037.575
01/01/10–01/08/12	31	100	5%	$3301.786	$22,887.846	$3301.7866

So by 1st August 2012, there is $22,887.84 in the account.