MATH 2600/STAT 2600, Theory of Interest
FALL 2010
Toby Kenney
Homework Sheet 5
Model Solutions

1. *A loan of \$4,000 at $j_{12} = 8\%$ is amortised with equal monthly payments for 6 months, with the first payment in one month.*

   (a) Calculate the monthly payments.

$$4000 = R\frac{1 - (1 + \frac{0.08}{12})^{-6}}{\frac{0.08}{12}}$$

$$R = \frac{4000 \times \frac{0.08}{12}}{1 - (1 + \frac{0.08}{12})^{-6}} = 682.308$$

   So the monthly payments are \$682.31.

   (b) Draw up a complete amortisation schedule for the loan

Payment	Interest	Principal	Outstanding Balance
			4000
682.31	26.67	655.64	3344.36
682.31	22.30	660.01	2684.34
682.31	17.90	664.41	2019.93
682.31	13.47	668.84	1351.08
682.31	9.01	673.30	677.78
682.30	4.52	677.78	0

2. *Mr. Thompson takes out a 25 year mortgage for a loan of \$250,000 at $j_2 = 7\%$. After 5 years, the interest rate increases to 9%.*

   (a) Calculate the outstanding balance after 5 years.

   The monthly payments R satisfy:

$$250000 = R\frac{1 - 1.035^{-50}}{1.035^{\frac{1}{6}} - 1}$$

$$R = 1751.039$$

   So the monthly payments are \$1751.04.

   Retrospective method:

   The accumulated value of the loan is $250000 \times 1.035^{10} = 352649.690$. The accumulated value of the first 5 years of payments is $1751.04\frac{1.035^{10}-1}{1.035^{\frac{1}{6}}-1} = 125038.257$, so the outstanding balance is the difference between these, or \$227,611.43

Prospective method:

We need to calculate the reduced final payment:

$$1751.04 \frac{1 - 1.035^{50}}{1.035^{\frac{1}{6}} - 1} = 250000.109...$$

, so the excess payment is \$0.109... at the time the mortgage is taken out, so the final payment is reduced by $0.109... \times 1.035^{50} = 0.61$. (That is, the final payment is \$1750.43.) The discounted value of the 61¢ reduction 5 years from the start of the mortgage is $0.61 \times 1.035^{-40} = 0.154....$ The discounted value of 20 years of monthly payments of \$1751.04 at $j_2 =$ 7% is $1751.04 \frac{1-1.035^{-40}}{1.035^{\frac{1}{6}}-1} = 227611.588...$, so the discounted value of the remaining payments is obtained by subtracting 0.154... from this, giving \$227,611.43.

(b) What should the new monthly payments be if he wishes to keep the term of the mortgage at 25 years?

He wants to repay the loan with another 20 years of payments, so we need to solve

$$227611.43 = R \frac{1 - 1.045^{-40}}{1.045^{\frac{1}{6}} - 1}$$
$$R = 2023.897$$

so the new monthly payments should be \$2023.90.

(c) If instead, he wishes to keep the monthly payments the same (or as near as possible with the mortgage a whole number of years):

(i) what will the new term of the mortgage be, and what will the monthly payments be?

He wants to keep the monthly payments at \$1751.04, to pay off the remaining \$227,611.43, at $j_2 = 9\%$. We therefore need to solve

$$227611.43 = 1751.04 \frac{1 - 1.045^{2n}}{1.045^{\frac{1}{6}} - 1}$$

$$1.045^{-2n} = 1 - \frac{227611.43}{1751.04} \times (1.045^{\frac{1}{6}} - 1)$$

$2n = 71.54...$, so the mortgage should be refinanced over another 36 years (for a total of 41 years).

The new monthly payments are obtained by solving

$$227611.43 = R \frac{1 - 1.045^{72}}{1.045^{\frac{1}{6}} - 1}$$
$$R = 1749.472...$$

so the monthly payments are \$1749.48.

(ii) How long must he wait before the payments after the interest rate change have as large a principal component as the last payment before the interest rate change?

If the interest rate had not changed, the monthly payment would have covered interest of $227611.43(1.035^{\frac{1}{6}} - 1) = 1308.77...$, so the principal component would have been $1751.04 - 1308.77 = 442.26....$ The payment before this, the principal component was smaller by a factor of $1.035^{\frac{1}{6}}$, so it was \$439.74.

The first payment after the change has an interest component of $227611.43(1.045^{\frac{1}{6}} - 1) = 1675.931$, so the principal component is $1749.48 - 1675.931 = 73.549....$ With each payment, the principal component increases by a factor of $1.045^{\frac{1}{6}}$, so to find out how long before it reaches \$439.74, we need to calculate $\frac{\log \frac{439.74}{73.549...}}{2 \log 1.045} = 20.31$ years, or about 20 years 4 months.

3. *Mr. and Mrs. Johnson buy a house for \$500,000 with a downpayment of \$100,000 and a 20 year mortgage for the remaining \$400,000 at a rate $j_2 = 6\%$. This results in monthly payments of \$2848.76. The mortgage allows them to pay the balance off early for a penalty of three times the monthly interest on the outstanding balance.*

   After 7 years, at what interest rate should they be willing to take out a loan in order to pay off the remaining balance (including the penalty)?

   After 7 years, the remaining balance is given by

$$400000 \times 1.03^{14} - 2848.76\frac{1.03^{14} - 1}{1.03^{\frac{1}{6}} - 1} = 309357.24...$$

   The 3 months interest penalty is therefore $309357.24... \times 3 \times (1.03^{\frac{1}{6}} - 1) = 4583.40$, so the amount they would need to pay off the loan would be \$313,940.64. They want to refinance this over 13 years, so that the monthly payments are at most 2848.76.

   That is, they need to find i such that $\frac{1-(1+i)^{-156}}{i} \geqslant \frac{313940.64}{2848.76} = 110.20...$

   For $i = 0.01$, $\frac{1-(1+i)^{-156}}{i} = 78.82$ For $i = 0.005$, $\frac{1-(1+i)^{-156}}{i} = 108.14$ For $i = 0.004$, $\frac{1-(1+i)^{-156}}{i} = 115.88$

   Now we interpolate. $\frac{110.20-108.14}{115.88-108.14} = 0.2663...$, so by interpolation we get $i = 0.0047337...$, which gives 110.12..., so they could afford to refinance at i=0.0047337, or $j_2 = 2(1.0047337^6 - 1) = 0.0575$, or 5.75%.

4. *Jim takes out a loan for \$15,000, to be repayed over 36 months at $j_{12} = 6\%$.*

   The monthly repayments are given by

$$15000 = R\frac{1 - 1.005^{-36}}{0.005}$$

$$R = 456.329..$$

So the monthly repayments are \$456.33.

Calculate the outstanding balance after 4 months using

(a) The amortisation method.

After 4 months, the original \$15,000 is worth $15000\times1.005^4 = 15,302.2575...$, while the payments are worth $456.33\frac{1.005^4-1}{0.005} = 1839.056...$, so the outstanding balance is \$13,463.20.

(b) The sum-of-digits method.

The total value of 36 payments of \$456.33 is $456.33\frac{1-1.005^{-36}}{0.005} = 15000.03...$, so we need to calculate the value of the excess 3 cents after 36 months. $0.03... \times 1.005^{36} = 0.04$, so the final payment is reduced by 4 cents.

The total amount of interest due to be paid is therefore $456.33 \times 36 - 15000.04 = 1427.84$ The first 4 payments should account for $\frac{36+35+34+33}{\frac{36\times37}{2}}$ of this interest, or \$295.86 of interest. Therefore, they cover $456.33 \times 4 - 295.96 = 1529.46$ of principal. This leaves a balance of \$13,470.54.

5. *Matthew takes out a mortgage for \$300,000 for 20 years at* $j_2 = 6\%$.

   (a) Calculate the monthly repayments.

$$300000 = R\frac{1-1.03^{-40}}{1.03^{\frac{1}{6}}-1}$$
$$R = 2136.565...$$

   So the monthly repayments are 2136.57.

   (b) The loan is due to be refinanced after 5 years, when Matthew will be allowed to make changes to his monthly payments. The bank charges a penalty of 3 months interest for refinancing before that time. After 1 year, Matthew gets a promotion, and can now afford payments of up to \$3,000 a month. If he can earn interest of $j_{12} = 3\%$ *on his savings, should he refinance the loan early?*

   After 1 year, the outstanding balance is $300000\times1.03^2 - 2136.57\frac{1.03^2-1}{1.03^{\frac{1}{6}}-1} = 291923.15$

   The penalty for refinancing is therefore $291923.15 \times 3 \times (1.03^{\frac{1}{6}} - 1) = 4325.09$, so to refinance, he has to pay off a loan of \$296,248.24 with monthly payments of \$3,000. The number of payments is therefore given by solving

$$296248.24 = 3000\frac{1-1.03^{-\frac{n}{6}}}{1.03^{\frac{1}{6}}-1}$$
$$1.03^{\frac{n}{6}} = 1 - \frac{296248.24}{3000}(1.03^{\frac{1}{6}}-1)$$

$$n = \frac{-6\log\left(1 - \frac{296248.24}{3000}(1.03^{\frac{1}{6}} - 1)\right)}{\log 1.03} = 135.76$$

On the other hand, if he doesn't refinance until after 5 years, he can save up the leftover \$863.53 every month at 3% interest. After 4 years, this will be worth $863.53\frac{1.0025^{48}-1}{0.0025} = 43980.63$. The outstanding value of the mortgage will be $300000 \times 1.03^{10} - 2136.57\frac{1.03^{10}-1}{1.03^{\frac{1}{6}}-1} = 254388.18$. If he now uses his savings to pay off some of this, it leaves \$210,407.55. To pay this off with monthly payments of \$3,000 would require n monthly payments where n is the solution to

$$210407.55 = 3000\frac{1 - 1.03^{-\frac{n}{6}}}{1.03^{\frac{1}{6}} - 1}$$

$$1.03^{\frac{n}{6}} = 1 - \frac{210407.55}{3000}(1.03^{\frac{1}{6}} - 1)$$

$$n = \frac{-6\log\left(1 - \frac{210407.55}{3000}(1.03^{\frac{1}{6}} - 1)\right)}{\log 1.03} = 86.31$$

more payments. This is in addition to the 48 payments already made, for a total of 134.31 payments. This is fewer payments than would be needed to refinance after 1 year, so it is better to wait until 5 years before refinancing.

6. *Mr. Robinson borrows \$1,000,000 at $j_{12} = 9\%$ to set up his business. He has two options for repaying the debt. He may either amortise the debt over 15 years, or he may pay off the interest each month, and set up a sinking fund to pay off the debt at the end of 15 years. The sinking fund will earn $j_{12} = 8.5\%$.*

   (a) Calculate the monthly payments needed for each of the options.

   For the amortisation method, the monthly payments are given by

$$1000000 = R\frac{1 - 1.0075^{180}}{0.0075}$$

$$R = 10142.665$$

   so the monthly payments are \$10,142.67.

   For the sinking fund method, the monthly interest payments are \$7,500, while the sinking fund needs to accumulate \$1,000,000 at $j_{12} = 8.5\%$ over 15 years, so we need to solve

$$R\frac{(1 + \frac{0.085}{12})^{180} - 1}{\frac{0.085}{12}} = 1000000$$

$$R = 2764.062...$$

so the monthly payments to the sinking fund are \$2,764.07, making the total monthly expense of the debt \$10,264.07.

(b) Mr. Robinson expects that he will want to invest a further \$200,000 into his business after 6 years. If he is paying off the loan using a sinking fund, he will just use the money in the sinking fund to pay it off. If he has taken the amortised loan, then he will have to take out another loan at $j_{12} = 12\%$ *to make this investment. Calculate the new monthly payments after 6 years in each case.*

After 6 years, the money in the sinking fund is

$$2764.07\frac{(1+\frac{0.085}{12})^{72}-1}{\frac{0.085}{12}} = 258443.83$$

. If he takes out \$200,000, this will leave \$58,443.83, which will grow to $58443.83(1+\frac{0.085}{12})^{108} = 125257.22$, so the remaining payments in the fund have to accumulate to the remaining \$874,742.78 after 9 years. For this, the sinking fund payments need to be

$$874742.78\frac{0.085}{12\times((1+\frac{0.085}{12})^{108}-1)} = 5419.923...$$

so the monthly payment to the sinking fund is \$5,419.93, for a total monthly payment of \$12,919.93.

On the other hand, if he takes out a loan at $j_{12} = 12\%$, for this \$200,000, then the monthly repayments will be

$$200000\frac{0.01}{1-1.01^{108}} = 3036.847$$

, so the monthly repayments will be \$3,036.85, in addition to the monthly repayments of \$10,142.67 from the original loan, for a total monthly repayment of \$13,179.52.