MATH 2600/STAT 2600, Theory of Interest
FALL 2010
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Homework Sheet 7
Model Solutions

1. *A machine has a current cost of \$40,000. The machine has an expected lifetime of 20 years. It cannot be resold when it is finished with. The maintainance costs are \$700 in the first year, and increase by \$500 in each subsequent year. The cost of capital is $j_1 = 6\%$.*

   (a) What is the total capitalised cost if the machine is replaced every 20 years?

   If it is replaced every 20 years, the equivalent value of the maintainance costs for the each machine at the time of purchase is

   $$\frac{700 + 500\frac{1-1.06^{-20}}{0.06} - (700 + 500 \times 20)1.06^{-20}}{0.06} = 51644.167...$$

   The replacement cost is \$40,000. The total capitalised cost is therefore $91644.167...(1 + \frac{1}{1.06^{20}-1}) = 133165.94$

   [Alternatively, we could calculate an equivalent maintenance cost at the end of the 20 years, and use our formula for a perpetuity, or we could calculate an equivalent annual maintenance cost.]

   (b) It turns out to be best to replace the machine every 15 years. What is the total capitalised cost in this case?

   If it is replaced every 15 years, the equivalent value of the maintainance costs for the each machine at the time of purchase is

   $$\frac{700 + 500\frac{1-1.06^{-15}}{0.06} - (700 + 500 \times 20)1.06^{-15}}{0.06} = 35575.85...$$

   The replacement cost is \$40,000. The total capitalised cost is therefore $75575.85...(1 + \frac{1}{1.06^{15}-1}) = 129691.64$

   (c) Another type of machine for the same task has a current cost of \$80,000, but it's price is expected to fall by 4% every year, as the technology improves. It's maintainance costs are \$2,000 a year. It also lasts 20 years. Would this machine be cheaper in the long run? [Retraining costs prevent buying the cheaper machine first, and then changing to the other machine when it becomes cheaper.]

   The total capitalised cost of this machine is $\frac{80000}{1-\left(\frac{0.96}{1.06}\right)^{20}} = 92,787.88...$ for replacement, and $\frac{2000}{0.06} = 33333.33...$ for maintainance, giving a total capitalised cost of \$126,121.22, so this machine is cheaper in the long run.

2. *A computer is bought for \$1500. It is expected to last for 3 years, after which it will have a value of \$300. Prepare a depreciation schedule using*

   (a) The straight-line method.

Year	Yearly Depreciation	Accumulated Depreciation	Book Value
0	0	0	\$1,500
1	\$400	\$400	\$1,100
2	\$400	\$800	\$700
3	\$400	\$1,200	\$300

   (b) The constant percentage method.

   We calculate d by $1500(1-d)^3 = 300$. This gives $(1-d)^3 = 0.2$, so $d = 41.5...\%$.

Year	Yearly Depreciation	Accumulated Depreciation	Book Value
0	0	0	\$1,500
1	\$622.79	\$622.79	\$877.21
2	\$364.22	\$987.01	\$512.99
3	\$212.99	\$1,200	\$300

3. *A mining company buys a mine which they estimate containes 5,000 tonnes of ore, for \$1,000,000. After the mining is finished, they expect that they will be able to sell the land for a net price (after restoration costs) of \$200,000. In the first 3 years, the company mines 1,500 tonnes of ore. What is the book value of the mine after 3 years?*

   The depletion base is $1000000 - 200000 = 800000$. This is over 5,000 tonnes, so the depletion per tonne is \$160. Therefore, after 1,500 tonnes have been removed, the depletion is $1500 \times 160 = 240000$. This leaves a book value of $1000000 - 240000 = 760000$.

4. *Two standard fair dice are rolled. What is the probability that the larger number is 4?*

   The rolls where the larger number is 4 are: $(1,4)$, $(2,4)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$ and $(4,4)$. This gives a total of 7 out of 36 equally likely rolls, for a probability of $\frac{7}{36}$.

   (b) What is the expected value of the larger number?

   The possibilities, with probabilities are:

larger roll	probability	value × probability
1	$\frac{1}{36}$	$\frac{1}{36}$
2	$\frac{3}{36}$	$\frac{6}{36}$
3	$\frac{5}{36}$	$\frac{15}{36}$
4	$\frac{7}{36}$	$\frac{28}{36}$
5	$\frac{9}{36}$	$\frac{45}{36}$
6	$\frac{11}{36}$	$\frac{66}{36}$

The expected value is the total of the last column, which is $\frac{161}{36} = 4.47....$

5. *Mr. Davis is about to retire. The total value of his pension plan is \$150,000, and it is invested at $j_1 = 6\%$. The probability of his dying this year is 1%, the probability of dying in any subsequent year is such that the overall probability of dying within n years is $\frac{n}{50}$ (i.e. the probability that he dies in year n is $\frac{1}{50-n}$).*

   This question was not well worded, because I was attempting to change it several times as the earlier version was too difficult. Some of the wording was ambiguous. I apologise for the confusion. The intended meaning was that the total probability of his dying in year n should be 2% for any of the next 50 years (and the conditional probability should be $\frac{1}{50-n}$). It's not totally clear what the actual wording means. I have given the solution for the intended meaning. It should be clear how it can be modified for other interpretations of the question.

   (a) What is the expected value of the remaining length of his life?

   The question doesn't make clear at what time in each year he is most likely to die, so it should probably be interpreted as the number of whole years that he survives [though other interpretations are also reasonable]. Now the expected value of the remaining length of his life has a $\frac{1}{50}$ probability of taking each of the values 0, 1, ..., 49. This gives an expected value of $\frac{1+2+\cdots+49}{50} = 24.5$ years.

   (b) If he decides to withdraw \$6,000 at the start of every year until he dies, what is the expected present value of all the withdrawls?

   The probability that he makes the nth withdrawl is $\frac{n}{50}$, so the expected present value of all the withdrawls is $P = 6000(1 + 1.06^{-1}\frac{49}{50} + \ldots + 1.06^{-49}\frac{1}{50})$. Now we have

$$\begin{aligned}0.06P = 1.06P - P &= 6000(0.06 + \frac{49}{50} - \frac{1}{50}(1.06^{-1} + 1.06^{-2} + \ldots + 1.06^{-48} + 1.06^{-49})\\ &= 6240 - 120\frac{1 - 1.06^{-50}}{0.06}\\ P &= 72476.28\end{aligned}$$

(c) If he were to make exactly 25 withdrawls, what would the present value be? Why is this answer different from the answer in (b)?

The present value of 25 withdrawls is $6000 + 6000\frac{1-1.06^{-24}}{0.06} = 81302.15$. This is different from the answer in (b), because decreasing the number of withdrawls affects the present value more than increasing it — for example, the withdrawl in 20 years has a higher present value than the withdrawl in 30 years, so the reduction in the probability that he makes the 20th withdrawl affects the expected value more than the increase in probability that he makes the 30th withdrawl. Thus there is a net decrease in the expected present value, when the number of payments becomes more uncertain.

The fact that 25 is different from 24.5 is not responsible for most of the difference. Indeed the answer to (b) is less than the present value of 24 withdrawls.

(d) (i) How much should he withdraw every year, so that the expected present value is equal to \$150,000?

If he withdraws X every year, the expected present value is $X\frac{1.04 - \frac{1-1.06^{-50}}{3}}{0.06}$, so for the expected present value to be \$150,000, X must be \$12,417.87.

(ii) Would it be a good idea for him to withdraw this amount every year?

Probably not — the increase in his quality of life for the first 23 years by being able to spend more money would most likely be offset by the risk of being completely broke after that time.

(iii) Would it be reasonable for a large company to agree to pay him this amount (less its commission) every year until he dies, in exchange for the money in his pension plan?

For the large company, making a loss on this money would not be such a problem — they can make up the loss on other deals. Put another way, a large company might make similar deals with 1000 people or more. In such a case, their overall loss would be much smaller, and would almost certainly less than their commissions, meaning they would be almost certain to make a profit. Therefore it would be reasonable for the large company to agree to pay him this amount.