MATH 2600/STAT 2600, Theory of Interest
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Homework Sheet 2
Model Solutions

1. *Mrs. Hall makes a loan of \$35,000 at $j_{12} = 4\%$. The loan is repaid over 5 years with equal monthly payments. When Mrs. Hall receives each payment, she immediately deposits it in an account which receives $j_{12} = 3\%$ interest. What yield does she make on her investment at the end of the 5 years?*

   The monthly payments are given by $\frac{35000}{a_{\overline{60}|0.0033333}} = \frac{35000\times 0.0033333333}{1-1.003333333^{-60}} = 644.58$. The accumulated value of these payments at the end of 5 years is $644.58 s_{\overline{60}|0.0025} = 644.58\frac{1.0025^{60}-1}{0.0025} = 41669.87$. Her annual effective yield is therefore $\left(\frac{41669.87}{35000}\right)^{\frac{1}{5}} - 1 = 3.55\%$.

2. *Mr. Izzard invests \$500 a month at $j_{12} = 4\%$ into a fund for his retirement.*

   (a) How much is in the fund when he retires 14 years 3 months from the first investment in the fund?

   Including the final payment when he retires, he makes 172 payments into the fund. The amount in the fund at the end is therefore $500 s_{\overline{172}|0.0033333333} = 500\frac{1.00333333-1}{0.00333333333} = \$115,872.05$.

   (b) If he needs to accumumlate \$150,000 before retiring, how long does he have to delay his retirement?

   We want $500 s_{\overline{n}|0.0033333333} = 150000$, or $s_{\overline{n}|0.0033333333} = 300$, i.e. $\frac{1.0033333333^n-1}{0.00333333} = 300$, which gives $1.0033333333^n = 2$, so $n = \frac{\log(2)}{\log(1.003333333)} = 208.29$ months, so he has to delay his retirement by 36.29 months, or just over 3 years.

3. *Mrs. Jacobs wants to save up \$100,000 for her son's education. She starts a savings account, which pays interest at $j_{12} = 4\%$. She plans to make monthly contributions into the account from now until her son starts university in 9 years 5 months time.*

   (a) How much should she pay into the account each month in order to meet her savings goal?

   She can make 114 payments into the account. The amount she needs to pay monthly is $\frac{100000}{s_{\overline{114}|0.00333333}} = \frac{100000\times 0.0033333333}{1.00333333333^{114}-1} = \$722.50$.

   (b) If she can only afford to contribute \$650 each month, what interest rate does she need to receive on her savings?

We need to solve $s_{\overline{114}|i} = \frac{100000}{650} = 153.846$. We try a variety of interest rates i.

i	$s_{\overline{114}\|i}$
0.005	153.151
0.006	162.956
0.0051	154.098
0.00507	153.813
0.00508	153.908
0.005071	153.823
0.005079	153.899

so the interest is between $j_{12} = 6.0852\%$ and $j_{12} = 6.0948\%$, so she needs to invest at $j_{12} = 6.09\%$.

4. *Mr. Keith borrows \$50,000 from a bank at $j_{12} = 8\%$. He plans to pay this off with monthly payments over the next 4 years.*

   (a) What are the monthly payments?

   The monthly payments are given by $\frac{50000}{a_{\overline{48}|0.0066666667}} = \frac{50000\times 0.0066666667}{1-1.0066666667^{48}} = \$1,220.65$.

   (b) After 2 years, the bank sells the loan to another bank, which wants to receive $j_{12} = 6.7\%$. How much does that bank pay for the loan?

   The second bank will receive 24 monthly payments of \$1,220.65, which at $j_{12} = 6.7\%$ have a present value of $1220.65a_{\overline{24}|0.0055833333333} = \$27,346.24$.

5. *A company is investing into its pension plan. It makes monthly payments of \$300,000 into an account that pays $j_{12} = 6\%$, starting in January 2003. In January 2006, interest rates drop to $j_{12} = 5\%$. In September 2007, one employee retires, and the company decreases the net monthly payments starting with the September payment to \$280,000. How much is in the account when the pension plan is audited immediately following the payment in March 2010?*

   After it makes the January 2006 payment, the amount in the plan is $300000a_{\overline{37}|0.005} = \$10,110,751.12$. By September 2007, this has grown to $10110751.12(1.00416666667)^{20}$.

6. *Mr. and Mrs. Lewis are retiring. They have saved up \$1,600,000, from which they want to live for the next 35 years. They want to take out monthly withdrawls, which will increase every month in line with inflation at an annual rate of 2%. (That is, the withdrawls form a geometric progression, with each payment 2% more than the one twelve months earlier.) If their money is invested at $j_{12} = 5\%$, how much should the first withdrawl be?*

   The effective monthly rate of interest is $\frac{1+\frac{0.05}{12}}{1.02^{\frac{1}{12}}} - 1 = 0.2510938\%$. The zeroth withdrawl would therefore by $\frac{1600000}{a_{\overline{420}|0.002510938}} = 6169.33$. The first withdrawl would then be $6169.33(1.02)^{\frac{1}{12}} = \$6,179.52$.

7. *Mr. Monroe donates \$5,000,000 to his old university. He states that the donation should be used to fund an annual scholarship of \$10,000 for each of 12 students. At what interest rate does the money need to be invested to provide this scholarship forever.*

   Each year, the fund needs to pay out \$120,000, so we have $5000000i = 120000$, so $i = \frac{120000}{5000000} = 2.4\%$.

8. *Mr. Neville deposits \$1,200 every quarter into an account which pays interest at $j_{12} = 4\%$. How much is in the account when he makes the 17th deposit?*

   The quarterly interest rate is $1.0033333333^3 - 1 = 1.0033\%$, so the amount in the account after the 17th deposit is $1200s_{\overline{17}|0.010033} = \$22,122.55$.

9. *Mrs. Owen is saving up to go on holiday. Every day she puts \$25 into an account which pays interest at $j_1 = 4\%$. How long does she have to wait before she has saved up \$6,500 for her holiday?*

   The daily interest rate is $1.04^{\frac{1}{365}} = 0.010746\%$, so the amount she has saved after n days is $25\frac{1.00010746^n-1}{0.00010746}$. We solve $25\frac{1.00010746^n-1}{0.00010746} = 6500$, which gives $1.00010746^n = 1 + 260 \times 0.00010746 = 1.02793960$. We therefore get $n = \frac{\log(1.02793960)}{\log(1.00010746)} = 256.45$ days, so she has to wait 257 days to save up enough money.

10. *Dr. Parker takes out a loan for \$14,000 at $j_{12} = 6\%$. He wants to repay the loan with an increasing arithmetic progression of payments. He would like the first payment in one month's time to be \$100, and he would like the loan to be payed off after three years. By how much should the payments increase each month?*

    36 payments of \$100 have a present value of $100a_{\overline{36}|0.005} = 100\frac{1-1.005^{-36}}{0.005} = \$3,287.10$. The increasing payments therefore have a present value of \$10,712.90. Let X be the increase in the payments. The present value of these increasing payments is $P = X1.005^{-2} + 2X1.005^{-3} + \cdots + 35X1.005^{-36}$, so we have $0.005P = 1.005P - P = X1.005^{-1} + X1.005^{-2} + \cdots + X1.005^{-35} - 35X1.005^{-36} = X(a_{\overline{35}|0.005} - 35(1.005)^{-36})$, so $X = \frac{0.005\times 10712.90}{a_{\overline{35}|0.005} - 35(1.005)^{-36}} = \frac{53.56}{\frac{1-1.005^{-35}}{0.005} - 35(1.005)^{-36}} = \$19.21$.

11. *The stock of company ABC currently pays a dividend of \$0.70 every quarter. Every year the company increases the dividend by 3%. What is a fair price for the stock at $j_{12} = 8\%$?*

    At $j_{12} = 8\%$, the value of all the first years dividends at the end of the year is $0.70s_{\overline{4}|1.006666666666667^3-1} = 0.70\frac{1.006666666666667^{12}-1}{1.006666666666667^3-1} = \$2.89$. The zeroth years dividends should therefore have a value $\frac{2.89}{1.03} = \$2.80$. The real effective annual rate of interest is $\frac{1.0066666666666667^{12}}{1.03} - 1 = 5.14\%$, so the value of the stock is $\frac{2.80}{0.0514} = \$54.45$.

12. *A company buys a machine for \$90,000. The machine is expected to last for 6 years, after which it will have a salvage value of \$12,000. Prepare a depreciation schedule using:*

Total Depreciation$=90000-12000=\$78,000$.

(a) The sum of digits method.

Year	Value at start of year	Depreciation
0	\$90,000	\$22,285.71
1	\$67,714.29	\$18,571.43
2	\$49,142.86	\$14,857.14
3	\$34,285.72	\$11,142.86
4	\$23,142.86	\$7,428.57
5	\$15,714.29	\$3,714.29
6	\$12,000	

(b) The constant percentage method

$\left(\frac{12000}{90000}\right)^{\frac{1}{6}}=0.714753772,$

Year	Value at start of year	Depreciation
0	\$90,000	\$25,672.16
1	\$64,327.84	\$18,349.27
2	\$45,978.57	\$13,115.22
3	\$32,863.35	\$9,374.14
4	\$23,489.21	\$6,700.21
5	\$16,789.00	\$4,789.00
6	\$12,000	

(c) The straight line method

Year	Value at start of year	Depreciation
0	\$90,000	\$13,000
1	\$77,000	\$13,000
2	\$64,000	\$13,000
3	\$51,000	\$13,000
4	\$38,000	\$13,000
5	\$25,000	\$13,000
6	\$12,000	

(d) The compound interest method, with cost of capital $j_1=5\%$

Sinking fund payments $\frac{78000}{s_{\overline{6}|0.05}}=\frac{78000\times 0.05}{1.05^6-1}=\$11,467.36$

Year	Value at start of year	Depreciation
0	\$90,000	\$11,467.36
1	\$78,532.64	\$12,040.73
2	\$66,491.91	\$12,642.77
3	\$53,849.14	\$13,274.91
4	\$40,574.23	\$13,938.65
5	\$26,635.58	\$14,635.58
6	\$12,000	

13. *A company is deciding between two printers. The first printer costs \$1,600, lasts for 3 years, after which it has a resale value of \$200, and has ink and maintainance costs of \$130 in the first year, and increasing by \$20 every subsequent year. The second printer costs \$2,300, lasts for 4 years, with a resale value of \$300, and has ink and maintainance costs of \$80 in the first year, and increasing by \$10 in each subsequent year.*

    (a) If the cost of capital is $j_1 = 8\%$, which printer has lower total capitalised cost?

    The value of the ink and maintainance costs at the time of buying the printer for the first printer are $130(1.08)^{-1} + 150(1.08)^{-2} + 170(1.08)^{-3} = 383.92$, so the total cost for the printer is \$1,983.92. The total present value of all replacement costs is therefore $\frac{1783.92}{1.08^3-1} = \$6,868.85$, plus \$1,983.92 for the first printer is \$8,852.77.

    The value of the ink and maintainance costs at the time of buying the printer for the second printer are $80(1.08)^{-1} + 90(1.08)^{-2} + 100(1.08)^{-3} + 110(1.08)^{-4} = 311.47$. The total cost per printer is therefore \$4,611.47. The total present value of all replacement costs is therefore $\frac{4311.47}{1.08^4-1} = \$6,412.04$, plus the original \$4,611.47 makes a total capitalised cost of \$16,571.53. Therefore, the first printer has a lower total capitalised cost.

    (b) Suppose the first printer will print 1,000 pages in a day, but the second printer will print 1,200 pages in a day. Which printer has the lower total capitalised cost per page?

    The total capitalised cost per page per day for the first printer is $\frac{8852.77}{1000} = \$8.85$, while the cost for the second printer is $\frac{16{,}571.53}{1200} = \$13.81$, so the first printer still has the lower total capitalised cost.

    (c) [bonus] At what cost of capital would the two printers have the same total capitalised cost per page?

    The total capitalised cost per page per day for the first printer is $\frac{1600+130(1+i)^{-1}+150(1+i)^{-2}+170(1+i)^{-3}+(1400}{}$ while the total capitalised cost per page per day for the second printer is $\frac{2300+80(1+i)^{-1}+90(1+i)^{-2}+100(1+i)^{-3}+110(1+i)^{-4}+(2000+80(1+i)^{-1}+90(1+i)^{-2}+100(1+i)^{-3}+110(1+i)^{-4})((1+i)^4-1)^{-1}}{1200}$.

    We evaluate these for a range of values:

j_1	Total Capitalised cost per page per day	
	Printer 1	Printer 2
8%	\$8.52	\$7.52
12%	\$6.29	\$5.47
16%	\$5.02	\$4.46
30%	\$3.26	\$3.07
50%	\$2.479	\$2.477
51%	\$2.457	\$2.461
50.4%	\$2.4699	\$2.4705
50.3%	\$2.47218	\$2.47219
50.29%	\$2.47240	\$2.47235
50.295%	\$2.47229	\$2.47227

So they have the same total capitalised cost per page per day at cost of capital $j_1 = 50.30\%$.