MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 5
Due: Friday 26th October: 3:30 PM

Basic Questions

1. Which of the following functions are homomorphisms?

   (a) $f : S_5 \longrightarrow S_3$ sending ϕ to the permutation obtained by restricting ϕ to $\{1, 2, 3\}$ and then relabelling the image of $\{1, 2, 3\}$ as $\{1, 2, 3\}$ in order. For example, if $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$, then the image of $\{1, 2, 3\}$ is $\{2, 4, 5\}$, so we relabel in order $2 \mapsto 1$, $4 \mapsto 2$ and $5 \mapsto 3$. This gives $f(\phi) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

   (b) $f(x) = \pi x$ from $\mathbb{R}$ to itself (with $+$ as the group operation).

   (c) Let G be the group of 2×2 upper triangular real matrices with non-zero diagonal entries. Let $f : G \longrightarrow \mathbb{R}^*$ be the function sending a matrix in G to its bottom-right element.

   (d) $f(x) = e^x$ from $\mathbb{R}$ with $+$ as the group operation to $\mathbb{R}^*$ with multiplication as the group operation.

2. Which of the following subgroups are normal?

   (a) The rational numbers as a subgroup of the real numbers.

   (b) The subgroup of S_4 generated by $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$.

   (c) The group of complex matrices X for which $\det(X)^{34} = 1$. [Hint: recall from linear algebra that $\det(AB) = \det(A)\det(B)$.]

3. Find the kernel and image of the following homomorphisms.

   (a) Let G be the the group of symmetries of a cube. Define $f : G \longrightarrow S_4$ by the induced permutation on the diagonals.

   (b)

4. Show that the function $G \xrightarrow{f} G$ given by $f(x) = x^2$ on a group G is a homomorphism if and only if G is abelian.

Theoretical Questions

5. Show that the composite of two homomorphisms is a homomorphism.

6. Show that a homomorphism of groups $G \xrightarrow{\phi} G'$ is an isomorphism if and only if there is a homomorphism $G' \xrightarrow{\phi'} G'$ such that the composites $\phi\phi'$ and $\phi'\phi$ are both the identity homomorphism.

7. Let $\sim$ be an equivalence relation on a group G such that whenever $x \sim x'$ and $y \sim y'$, we also have $xx' \sim yy'$.

   (a) Show that the subset $\{x \in G | x \sim e\}$, where e is the identity element of G, is a normal subgroup H.

   (b) Show that the equivalence relation $\sim$ is given by $x \sim y$ if and only if $xy^{-1} \in H$.

8. Show that any subgroup of index 2 is normal.

9. Show that if $H \leqslant G$ and N is a normal subgroup of G, then $N \cap H$ is a normal subgroup of H.

10. (a) Show that the intersection of two normal subgroups is another normal subgroup.

    (b) Show that the subgroup generated by two normal subgroups is normal.