MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 2
Due: Wednesday 3rd October: 3:30 PM

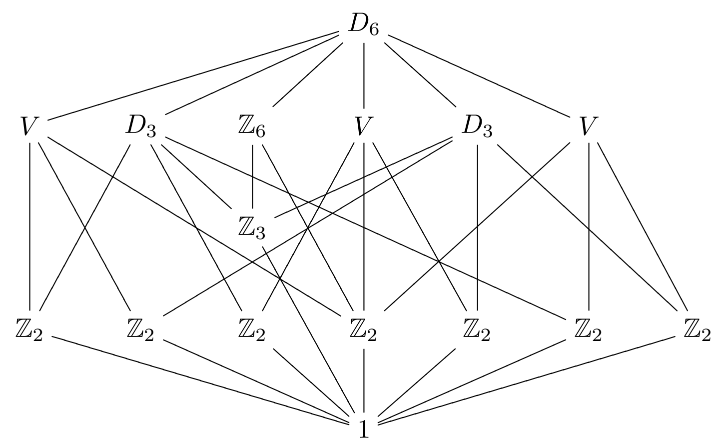
## Basic Questions

1. *(a) Show that the collection of symmetries of a regular hexagon is a group of order 12.*

The symmetries of a regular hexagon consist of 6 reflections — 3 perpendicular to the edges, and 3 in the diagonals, the identity and 5 rotations about the centre of the hexagon. This is a total of 12 elements. There is an identity. It is easy to see that reflections are their own inverses, and that rotation by $60n^\circ$ is inverse to rotation by $60(6-n)^\circ$, so it just remains to check associativity. However, a symmetry is a function from the plane to itself, and the group operation is composition of functions, so it is associative.

*(b) Find all subgroups of this group.*

There are many ways to solve this; one way is to look at the reflections in the subgroups. If a subgroup contains a $60^\circ$ rotation and a reflection, then it must be the whole of $D_6$. If it contains two reflections with an angle of $30^\circ$ between them, then they compose to give a $60^\circ$ rotation. The only way to have three reflections without any two having an angle of $30^\circ$ between them is to have either the three diagonal reflections or the three edge reflections. Either of these sets of three reflections generates a subgroup of symmetries of a triangle (the two sets generate symmetry groups of different triangles). if a subgroup contains only two reflections, then the reflections must be perpendicular, and the subgroup will consist of those reflections, a $180^\circ$ rotation and the identity. There are three pairs of perpendicular reflections, which give three different subgroups. The only subgroups containing only one reflection are the subgroups containing a reflection and the identity only. There are 6 such subgroups. Finally, the subgroups containing no reflections are subgroups of the group of rotations, which is cyclic of order 6, so its non-trivial subgroups are generated by rotations by $120^\circ$ and $180^\circ$ respectively. In summary, the subgroup diagram looks like




2. *How many elements are in the subgroup of $\mathbb{Z}_{45}$ generated by 12?*

   The subgroup generated by 12 consists of the following elements:

   $$\{0, 12, 24, 36, 3, 15, 27, 39, 6, 18, 30, 42, 9, 21, 33\}$$

   so there are 15 elements.

   Alternatively, the subgroup generated by 12 consists of all multiples of 3, and there are 15 multiples of 3.

3. *Which of the following are subgroups of the group of permutations of the 6-element set $\{1, 2, 3, 4, 5, 6\}$.*

   *(a) The collection of permutations that fix the subsets $\{1, 2, 3\}$ and $\{4, 5\}$.*

   Let $\phi$ and $\theta$ be permutations in $S_6$ that fix the sets $\{1, 2, 3\}$ and $\{4, 5\}$. We need to show that $\phi \circ \theta$ and $\phi^{-1}$ also fix the sets $\{1, 2, 3\}$ and $\{4, 5\}$. For $\phi \circ \theta$, if $x \in \{1, 2, 3\}$ then we know that $\theta(x) \in \{1, 2, 3\}$, so $\phi(\theta(x)) \in \{1, 2, 3\}$. Similarly, if $x \in \{4, 5\}$, then $\phi(\theta(x)) \in \{4, 5\}$. Similarly, if $x \in \{1, 2, 3\}$, we want to show that $\phi^{-1}(x) \in \{1, 2, 3\}$. Since $\phi$ preserves the set $\{1, 2, 3\}$, we have that $\{\phi(1), \phi(2), \phi(3)\}$ is a subset of $\{1, 2, 3\}$, and it has 3 elements, so it must be $\{1, 2, 3\}$. However, since $\phi$ is a bijection, if we have that $\phi(x) \in \{\phi(1), \phi(2), \phi(3)\}$, we must have $x \in \{1, 2, 3\}$.

   *(b) The collection of permutations that send the subset $\{1, 2\}$ to the subset $\{4, 5\}$.*

   This is not a subgroup. For example, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 2 & 3 & 6 \end{pmatrix}$ is in this set, but its square is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 1 & 6 \end{pmatrix}$, which is not in the set.

4. *Which of the following are subgroups of the additive group of real numbers:*

   *(a) The collection of real numbers greater than or equal to 0.*

   This is not a subgroup because it does not contain inverses of its elements.

*(b) The collection of numbers whose decimal expansion terminates after a finite number of decimal places. [Technically, these numbers have two decimal expansions and only one terminates after a finite number of places.]*

This is a subgroup — if $x$ terminates after $m$ decimal places and $y$ terminates after $n$ decimal places, then $x + y$ terminates after $\max(m, n)$ decimal places, and $-x$ terminates after $m$ decimal places.

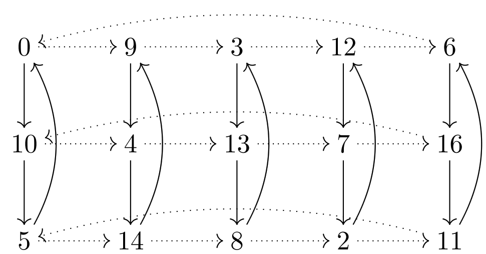
*(c) The collection of numbers $x$ such that $x^2$ is a rational number.*

This is not a subgroup, since it contains the elements 1 and $\sqrt{2}$, but $1+\sqrt{2}$ is not in the set, since $(1+\sqrt{2})^2 = 2+2\sqrt{2}$, which is not a rational number.

5. *How many generators does the cyclic group of order 28 have?*

   The cyclic group of order 28 is generated by all elements coprime to 28 — that is by all odd numbers not divisible by 7. There are 12 such numbers. They are $\{1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, 27\}$.

6. *Draw the Cayley graph of $\mathbb{Z}_{15}$ with generators 9 and 10.*




## Standard Questions

7. *Show that if a subgroup of the real numbers contains an interval $[a, b]$, with $a < b$ then it must be the whole group of real numbers.*

   Let $G$ be a subgroup of the real numbers containing the interval $[a, b]$. Let $x$ be a real number. Since $G$ is a subgroup of $\mathbb{R}$, $x$ is in $G$ if and only if $-x$ is, so assume without loss of generality that $x$ is positive. Since $a < b$, we must have $b - a > 0$, so $\frac{x}{b-a}$ is finite, and we can choose an integer $n > \frac{x}{b-a}$. Now since $a \in G$ and $a+\frac{x}{n} \in G$, we have that $\frac{x}{n} = a+\frac{x}{n}-a \in G$. Now by repeated addition, we have that $x = \frac{x}{n} + \frac{x}{n} + \cdots + \frac{x}{n} \in G$.

8. *$H$ and $K$ are subgroups of $G$. The union $H \cup K$ is also a subgroup of $G$. Prove that $H \subseteq K$ or $K \subseteq H$.*

   Suppose that this is not the case, then we must have some elements $x$ and $y$ in $G$ with $x \in H$ but $x \notin K$ and $y \in K$ but $y \notin H$. Since $H \cup K$ is a subgroup, and we have $x \in H \cup K$ and $y \in H \cup K$, we must also have $z = xy \in H \cup K$. However, if $z \in H$, then $y = x^{-1}z \in H$, contradicting our assumption. On the other hand, if $z \in K$, then $x = zy^{-1} \in K$, contradicting our assumption. So we have shown that we cannot find such elements $x$ and $y$. This means we must have either $H \subseteq K$ or $K \subseteq H$.

9. *Show that a group with only finitely many subgroups is finite. [Hint: consider the cyclic subgroups generated by each element.]*

   Let $G$ be a group with only finitely many subgroups. Every element of $G$ is contained in a cyclic subgroup of $G$, so $G$ is a union of all its cyclic subgroups. $G$ cannot have an infinite cyclic subgroup, since this subgroup would be isomorphic to $\mathbb{Z}$, which has infinitely many subgroups, which would all be subgroups of $G$. Therefore, all the cyclic subgroups of $G$ are finite, so $G$ is a union of a finite set of finite sets, so it is finite.

10. *The centre $Z$ of a group $G$ is the set of all elements in $G$ that commute with all elements in $G$. That is $Z = \{a \in G | (\forall x \in G)(ax = xa)\}$. Prove that $Z$ is a subgroup of $G$.*

    We need to show that if $x$ and $y$ are in $Z$, then so are $xy$ and $x^{-1}$. For any element $a \in G$, we have $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$, so that $xy \in Z$. Similarly, $x^{-1}a = x^{-1}axx^{-1} = x^{-1}xax^{-1} = ax^{-1}$, so that $x^{-1} \in Z$.

11. *(a) If $G$ is a group, and every finitely generated subgroup of $G$ is cyclic, show that $G$ is abelian.*

    Let $x$ and $y$ be elements of $G$. The subgroup $< x, y >$ is cyclic, so it is abelian. Therefore in this subgroup, $xy = yx$, and this must also hold in $G$.

    *(b) must $G$ be cyclic?*

    No. Every finitely generated subgroup of the rational numbers is cyclic, but the rational numbers is not cyclic.