MATH/STAT 3360, Probability
FALL 2011
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Sample Final Examination

This Sample Examination has more questions than the actual exam, in order to cover a wider range of questions. The material from Chapters 1–5 is only partially covered here. You should look at the questions on the Sample Midterm Examination and the Midterm Examination to see the range of possible questions on those topics.

1. (a) A truncated standard normal random variable has probability density function

$$f_X(x) = \begin{cases} \frac{e^{-\frac{x^2}{2}}}{\int_a^b e^{-\frac{x^2}{2}}} & \text{if } a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

   That is, it is a normal distribution conditional on lying in the interval $[a, b]$.

   What is the expectation of a truncated standard normal distribution?

   (b) An investment advisor tells you that the amount by which the market will increase next year is normally distributed with mean 10% and standard deviation 10%. The bank offers an investment product which pays the value of the market, minus a 2% fee if the market increases by between 2% and 32%. If the market rises by less than 2%, then it pays out your original investment; if the market rises by more than 32%, it pays out your original investment plus 30%. What is the expected value of this investment? [Hint: divide into the three cases where the market increases by less than 2%, the market increases by between 2% and 32%, and the market increases by over 32%.]

2. If X is uniformly distributed on [-1,4], and Y is uniformly distributed on the interval $[0, 3]$, find the moment generating function of $X - Y$.

3. An investor's annual profit has an expectation of \$2,000 and a variance of 100,000,000. (Profits in different years are independent and identically distributed.)

   (a) After n years, for large n, what is the approximate distribution of the investors average annual profit?

   (b) After how many years is the probability that the investor has made an overall loss during those years less than 0.001?

   (c) If at any point, the investor has made a total loss of over \$100,000, he must pay a fine. After how many years is the danger that he will have to pay this fine greatest, and what is the danger after this many years [assuming the approximation from (a) is valid]?

(d) Using the one-sided Chebyshev inequality, what is the largest probability that he will have to pay the fine after n years.

(e) For what number of years is this maximised, and what is the upper bound on the probability in this case?

4. In a class of 100 students, the professor wants to determine what proportion of the students can answer a simple calculus question. The professor decides to test a random sample of 10 students. In fact 35 of the students could answer the question.

   (a) What is the expected number of students sampled who answer the question correctly?

   (b) What is the variance of the number of correct answers if the students are sampled with replacement, i.e. one student could be tested more than once?

   (c) What is the variance of the number of correct answers if the 10 students sampled must be 10 different students?

5. If X is exponentially distributed with parameter λ_1 and Y is exponentially distributed with parameter λ_2, what is the probability density function of $X + Y$? [Assume $\lambda_1 \neq \lambda_2$.]

6. X is normally distributed with mean 3 and standard deviation 3 and Y is normally distributed with mean 7 and standard deviation 4. What is $P(Y > X)$?

7. Let X have an exponential distribution with parameter λ. Suppose that given $X = x$, Y is normally distributed with mean x and variance σ^2.

   (a) What is the joint density function of X and Y?

   (b) What is the covariance of X and Y?

   (c) Conditional on $Y = 3$, what is the density function of X? [You do not need to calculate the constant factor.]

8. On average, in a certain course, 20% of students get an 'A' and 15% students get a 'B'. There are 25 students taking the course one year. What is the probability that 5 students get an 'A' and 5 students get a 'B'?

9. (a) How many different ways are there to distribute numbers 1–10 among 20 people in such a way that each number is used exactly twice? [Hint: order the people in any way, then the different ways correspond to sequences of the numbers 1–10 such that each number occurs twice.]

   (b) How many ways are there to divide 20 people into 10 pairs if the order of the pairs doesn't matter? [Hint: how many ways are there to divide 20 people into 10 pairs if the pairs are numbered 1–10?]

10. The number of claims an insurance company needs to pay follows a Poisson distribution with parameter 4.5. What is the probability that the company needs to pay out 5 or more claims?

11. The number of visitors to a particular website on a given day is approximately normally distributed with mean 12000 and variance 2000^2. A company is considering placing an advertisement on this website. It predicts that each visitor to the website will order its product with probability 0.02, and that all visitors to the website act independently.

    (a) What is the expectation and variance of the number of orders the company receives?

    (b) Using a normal approximation, what is the probability that the number of orders the company receives is more than 300?

    (c) If 300 is the maximum number of orders that the company is able to satisfy (that is, if the company receives over 300 orders, it sells only 300 products) what is the expected number of products sold?

12. A patient is given a routine test for a rare disease. The disease affects 4 people in 1000. The test is 98% accurate, so there is a 2% chance of giving the wrong result. The test result is positive (i.e. indicates the patient has the disease). What is the probability that the patient actually has the disease?

13. A gambler starts with \$2, and continues to make a series of \$1 bets with probability $\frac{1}{2}$ until he loses all his money, or until he has \$5. What is the expected number of bets until he has lost all his money or reached \$5? [Hint: let e_n be the expected number of bets conditional on having $\$n$, and condition on the outcome of the next bet to get a relation between the e_n.]

14. Random variables X and Y have joint density function $f_{X,Y}(x, y) = \frac{1}{\sqrt{10\pi}} e^{-\frac{5x^2+5y^2-6xy}{10}}$.

    (a) What is the joint density function of $Z = X + Y$ and $W = X - Y$?

    (b) Are W and Z independent?

    (c) What is $\mathrm{Cov}(X, Y)$? [Hint: consider $\mathrm{Var}(Z) - \mathrm{Var}(W)$.]

15. Let X and Y are independent normal random variables with mean 0 and variances σ_1^2 and σ_2^2 respectively.

    (a) What is the joint density function of X and Y?

    (b) If (R, Θ) are the polar coordinates of (X, Y) (that is, $R = \sqrt{X^2 + Y^2}$ and Θ is the solution to $R \cos \Theta = X$ and $R \sin \Theta = Y$) then what is the joint density function of R and Θ?

    (c) Calculate the marginal density function of Θ.