ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science)
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Homework Sheet 4
Model Solutions

Basic Questions

1. *Let X follow a negative binomial distribution with $r = 4$ and $\beta = 1.2$. What is the probability that $X = 8$?*

   $P(X = 8) = \binom{11}{8}\left(\frac{1}{2.2}\right)^8\left(\frac{1.2}{2.2}\right)^4 = .0266156303$

2. *The number of claims on each insurance policy over a given time period is observed as follows:*

Number of claims	*Number of policies*
0	*736*
1	*382*
2	*74*
3	*7*
4	*2*
5 or more	*0*

   Which distribution(s) from the $(a, b, 0)$-class and $(a, b, 1)$-class appear most appropriate for modelling this data?

   We compute the ratio $\frac{kp_k}{p_{k-1}}$:

k	$\frac{kp_k}{p_{k-1}}$
1	0.5190217391
2	0.3874345549
3	0.2837837837
4	1.1428571428

   Here we see that these ratios are decreasing, which means that the negative binomial is not appropriate, and instead, either a Poisson or a binomial should be used.

3. *X follows an extended modified negative binomial distribution with $r = -0.8$ and $\beta = 2$, and $p_0 = 0.4$. What is $P(X = 7)$?*

   This is an $(a, b, 1)$-distribution with $a = \frac{2}{1+2}$ and $b = -(1 + 0.8)\frac{2}{1+2}$. We also know that $p_1^T = \frac{-0.8\times 2}{(1+2)^{1-0.8}-(1+2)} = 0.912060776$. For the zero-modified distribution, we therefore get $p_1^M = 0.6\times 0.912060776 = .5472364656$. We then use the recurrence relation

$$p_2 = \left(\frac{2}{3} - \frac{1.8}{3}\right) p_1 = 0.0364824310$$
$$p_3 = \left(\frac{2}{3} - \frac{3.6}{9}\right) p_2 = 0.0097286482$$
$$p_4 = \left(\frac{2}{3} - \frac{3.6}{12}\right) p_3 = 0.0035671710$$
$$p_5 = \left(\frac{2}{3} - \frac{3.6}{15}\right) p_4 = 0.0015219929$$
$$p_6 = \left(\frac{2}{3} - \frac{3.6}{18}\right) p_5 = 0.0007102633$$
$$p_7 = \left(\frac{2}{3} - \frac{3.6}{21}\right) p_6 = 0.0003517494$$

4. *Let X follow a compound Poisson-Negative binomial distribution with parameters $\lambda = 3.3$, $r = 4.8$ and $\beta = 2.3$. Calculate the conditional probability that $X = 7$ given that $X \leqslant 10$.*

We can calculate this using the recursive formula

$$g_k = \frac{1}{1 - af_0} \sum_{i=1}^{k} \left(a + \frac{bi}{k}\right) f_j g_{k-j}$$

We know that $f_k = \binom{k+3.8}{k} 0.3030303^k 0.69696969^{4.8}$, so we obtain the following values:

k	f_k
0	0.176777383
1	.2571307388
2	.2259633760
3	.1552071673
4	.0917133261
5	.0489137739
6	.0242098476
7	.0113188897
8	.0050592006
9	.0021803962
10	.0009118020
11	.0003717539

For the Poisson distribution, we have $a = 0$ and $b = \lambda$, so using the recurrence, if we set $g_0 = 1$, we get

$$g_k = \sum_{i=1}^{k} \left(\frac{3.3i}{k}\right) f_i g_{k-i}$$

k	g_k
0	1
1	0.8485314380
2	1.1056819413
3	1.2467404102
4	1.3053235122
5	1.3000408476
6	1.2447087146
7	1.1529437003
8	1.0380119
9	0.9116626
10	0.7833687
11	0.6601146

The total $g_0 + \ldots + g_{10} = 11.15365$, so the conditional probability that $X = 7$ given that $X \leqslant 10$ is $\frac{1.1529437003}{11.15365} = 0.1033692$.

5. *Let X follow a mixed negative binomial distribution with $\beta = 1.5$ and r following a gamma distribution with $\alpha = 2$ and $\theta = 4$. What is the probability that $X = 2$?*

   For a fixed value of r, the probability that $X = 2$ is given by

$$P(X = 2) = \frac{r(r+1)}{2} 0.4^2 0.6^r = 0.08r(r+1)e^{\log(0.6)r}$$

   The overall probability is given as the expected value of this over the distribution of r. That is

$$P(X = 2) = \frac{0.08 \int_0^\infty r(r+1)e^{(\log(0.6)-0.25)r} dr}{4^2\Gamma(2)}$$

   Let $\phi = \frac{1}{0.25-\log(0.6)} = 1.31436162$. We now have

$$P(X = 2) = \frac{0.08 \int_0^\infty (r^2+r)e^{-\frac{r}{\phi}} dr}{16} = 0.005\left(\phi^3\Gamma(3) + \phi^2\Gamma(2)\right) = 0.03134394$$

Standard Questions

6. *An insurance company estimates that the number of claims made by an individual in a year follows a Poisson distribution with parameter λ, where λ varies between individuals, following a gamma distribution with $\alpha = 3$ and $\theta = 0.05$.*

   (a) What is the probability that a randomly chosen individual makes 3 claims in a given year?

   The gamma mixture of Poisson random variables follows a negative binomial distribution with $r = \alpha = 3$ and $\beta = \theta = 0.05$. The probability that this is 3 is $\frac{3\times4\times5}{6}\left(\frac{0.05}{1.05}\right)^3\left(\frac{1}{1.05}\right)^3 = 0.000932769$.

(b) If an individual has made 3 claims in a given year, what is the probability that that individual makes 3 claims in the next year?

If an individual has a given value of λ, the probability that they make 3 claims in a year for two consecutive years is $\left(e^{-\lambda}\frac{\lambda^3}{6}\right)^2$. The probability of this for a randomly chosen individual is the expected value of this probability over the distribution of λ. That is

$$P = \int_0^\infty \left(e^{-\lambda}\frac{\lambda^3}{6}\right)^2 \frac{\lambda^2}{0.05^3} e^{-\frac{\lambda}{0.05}}\,d\lambda = \int_0^\infty \frac{\lambda^8}{36\times 0.05^3} e^{-22\lambda}\,d\lambda$$

We have that

$$\int_0^\infty \lambda^8 e^{-22\lambda}\,d\lambda = \frac{\Gamma(9)}{22^9}$$

so we get $P = \frac{\Gamma(9)}{22^9\times 0.05^6\times 36} = 0.000007422$. The probability that an individual who made 3 claims last year makes 3 claims again this year is $\frac{0.000007422}{0.000932769} = 0.007956954$.

7. *An insurance company models the number of claims X on a given policy using a distribution from the $(a, b, 1)$-class. The company wants its distribution to match the observed mean $\mathbb{E}(X) = 0.475$ and probability of zero $P(X = 0) = 0.738$, and also wants $P(X > 3) = 0.01$. From this, they calculate $P(X = 1) = 0.1120652294$. Under this model, what is the probability that an individual makes 4 claims in a year? [Hint: for a general member of the $(a, b, 1)-$class, we have $\mathbb{E}(X) = \frac{p_1+(a+b)(1-p_0)}{1-a}$ and $p_1^T = \frac{a+b}{(1-a)^{-1-\frac{b}{a}}-1}$.]*

The mean of a distribution from the $(a, b, 1)$-class is given by $\frac{p_1+(a+b)(1-p_0)}{1-a}$, so we have $\frac{p_1+(a+b)(1-p_0)}{1-a} = 0.475$. Substituting $p_0 = 0.738$ and $p_1 = 0.112$ gives us $0.112 + 0.262(a + b) = 0.475 - 0.475a$. We are also given $p_2 + p_3 = 0.99-0.738-0.112 = 0.140$, so we have $0.112\left(\left(a+\frac{b}{2}\right)+\left(a+\frac{b}{2}\right)\left(a+\frac{b}{3}\right)\right) = 0.140$. We solve these equations

$$\begin{aligned}(2a + b)(3 + 3a + b) &= 7.5\\ 0.737a + 0.262b &= 0.363\end{aligned}$$

From this, we deduce:

$$\begin{aligned}b &= 1.3855 - 2.8130a\\ (1.3855 - 0.8130a)(4.3855 + 0.1870a) &= -7.5\\ 0.1520a^2 + 3.3062a - 6.0761 &= -7.5\end{aligned}$$

$a = -0.440$ or $a = -21.31$. $a = -0.440$ gives $b = 2.62$ and so $p_1 = \frac{2.62-0.440}{(1.440)^{\frac{2.62}{0.440}-1}-1} = 0.112$ as required. $a = -21.31$ gives $b = 61.3165$, and $p_1 = \frac{61.3165-21.31}{(22.31)^{\frac{61.3165}{21.31}-1}-1} = 0.0309$, so the first solution is the one required. That is $a = -0.440$, $b = 2.62$. [This is not actually a valid distribution. The closest valid distribution is $b = 6a = 2.64$, which is a binomial distribution with $n = 5$ and $p = \frac{0.44}{1-0.44} = .7857$.]

8. *An insurance company insures 200 houses. The number of claims resulting from these policies follows a compound Poisson-Binomial distribution with $\lambda = 12$, $n = 8$ and $p = 0.001$. The company's risk management division wants to ensure that the probability of receiving 2 or more claims should be at most 0.001. How many houses can the company insure while satisfying this condition?*

   (i) 52

   (ii) 88

   (iii) 147

   (iv) 260

   If the number of houses insured increases, the parameter λ will increase proportionally. The pgf of a Poisson is $e^{\lambda(z-1)}$, and for a binomial, it is $(1-p+pz)^n$. The pgf of the compound distribution is therefore $P(z) = e^{\lambda((1-p+pz)^n-1)} = e^{\lambda((0.999+0.001z)^8-1)}$. The derivatives of this are $P'(z) = 0.008\lambda(0.999+0.001z)^7 e^{\lambda((0.999+0.001z)^8-1)}$ Evaluating at zero gives $p_0 = P(0) = e^(-(1-0.999^8)\lambda) = e^{-0.007972056\lambda}$, and $p_1 = P'(0) = 0.008\lambda(0.999)^7 e^{\lambda(0.999^8-1)} = 0.007944168\lambda e^{-0.007972056\lambda}$. We want to choose λ so that the sum of these is 0.999. That is, we want $(1+0.007944168\lambda)e^{-0.007972056\lambda} = 0.999$.

   Evaluating this for the values of λ resulting from the options given, we get

   (i) $(1+0.007944168\times 3.12)e^{-0.007972056\times 3.12} = 0.99961088$

   (ii) $(1+0.007944168\times 5.28)e^{-0.007972056\times 5.28} = 0.998997405$

   (iii) $(1+0.007944168\times 8.82)e^{-0.007972056\times 8.82} = 0.99741161$

   (iv) $(1+0.007944168\times 15.6)e^{-0.007972056\times 15.6} = 0.992494833$

   So (ii) is the correct answer: they can insure up to 88 houses.

   [We can find an approximate answer by the Taylor approximation $e^{0.007972056\lambda} \approx 1+0.007972056\lambda+0.000031777\lambda^2$, which gives the equation $1+0.007944168\lambda = 0.999(1+0.007972056\lambda+0.000031777\lambda^2)$, or $0.001 = (0.007964084 - 0.007944168)\lambda + 0.000031745\lambda^2$, so $\lambda = 5.307649251$. This means the company can insure $\frac{5.307649251}{12} \times 200 = 88$ houses.]

Bonus Question

9. *Using the general recursion formula, show that the expected value of a distribution from the* $(a, b, 0)$*-class is given by* $\frac{a+b}{1-a}$.

We have $p_k = \left(a + \frac{b}{k}\right) p_{k-1}$. We therefore have $kp_k = akp_{k-1} + bp_{k-1}$, so

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} akp_{k-1} + \sum_{k=0}^{\infty} bp_k = a\sum_{l=0}^{\infty} lp_l + a\sum_{l=0}^{\infty} p_l + b\sum_{k=0}^{\infty} p_k = a + b + a\mathbb{E}(X)$$

We solve this to get

$$\mathbb{E}(X) = \frac{a+b}{1-a}$$

[For the $(a, b, 1)$-class,

$$\mathbb{E}(X) = p_1 + \sum_{k=2}^{\infty} kp_k = p_1 + \sum_{k=2}^{\infty} akp_{k-1} + \sum_{k=1}^{\infty} bp_k = p_1 + a\sum_{l=1}^{\infty} lp_l + a\sum_{l=1}^{\infty} p_l + b\sum_{k=1}^{\infty} p_k = p_1 + (a+b)(1-p_0) + a\mathbb{E}($$

We therefore get $\mathbb{E}(X) = \frac{p_1 + (a+b)(1-p_0)}{1-a}$.]