ACSC/STAT 3720, Life Contingencies I
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Homework Sheet 1
Model Solutions

Basic Questions

1. *An insurance company models the future lifetime of an individual as having survival function $S(x) = e^{-\left(\frac{x}{85}\right)^3}$. Calculate the force of mortality.*

Force of mortality is

$$\mu_x = -\frac{d}{dx}\log(S(x)) = \frac{d}{dx}\left(\frac{x}{85}\right)^3 = \frac{3x^2}{85^3}$$

2. *An insurance company models the future lifetime of an individual as having survival function $S(x) = e^{-\frac{x^2}{360}}$. Calculate:*

(a) The mean and standard deviation of T_x.

The mean of T_x is

$$\begin{aligned}\int_0^\infty S_x(t)\,dt &= \int_0^\infty e^{-\frac{(x+t)^2}{360}} e^{-\frac{x^2}{360}}\,dt \\ &= e^{\frac{x^2}{360}} \int_x^\infty e^{-\frac{y^2}{360}}\,dy \\ &= e^{\frac{x^2}{360}} \sqrt{360\pi}\left(1 - \Phi\left(\frac{x}{\sqrt{180}}\right)\right)\end{aligned}$$

The mean of ${T_x}^2$ is

$$\begin{aligned}2\int_0^\infty tS_x(t)\,dt &= 2e^{\frac{x^2}{360}}\int_0^\infty te^{-\frac{(x+t)^2}{360}}\,dt \\ &= 2e^{\frac{x^2}{360}}\int_x^\infty (y-x)e^{-\frac{y^2}{360}}\,dy \\ &= 2e^{\frac{x^2}{360}}\left(\left[-360e^{-\frac{y^2}{360}}\right]_x^\infty - x\sqrt{360\pi}\left(1-\Phi\left(\frac{x}{\sqrt{180}}\right)\right)\right) \\ &= 720 - 2xe^{\frac{x^2}{360}}\sqrt{360\pi}\left(1-\Phi\left(\frac{x}{\sqrt{180}}\right)\right)\end{aligned}$$

The variance of T_x is therefore

$$720 - 2xe^{\frac{x^2}{360}}\sqrt{360\pi}\left(1-\Phi\left(\frac{x}{\sqrt{180}}\right)\right) - \left(e^{\frac{x^2}{360}}\sqrt{360\pi}\left(1-\Phi\left(\frac{x}{\sqrt{180}}\right)\right)\right)^2$$

(b) The mean curtate future lifetime.

The probability of surviving for at least n years is $e^{-\frac{(x+n)^2-x^2}{360}} = e^{-\frac{2xn+n^2}{360}}$. The curtate expected future lifetime is therefore

$$\sum_{n=1}^{\infty} e^{-\frac{2xn+n^2}{360}}$$

3. *An insurance company uses a survival model with survival function ${}_tp_x = \left(1-\frac{t}{120-x}\right)^{\alpha}$. The company wants to ensure that under this model, an individual aged 60 has probability 0.5 of surviving for 20 years. What value of α should they choose?*

   The probability of an individual aged 60 surviving for 20 years is ${}_{20}p_{60} = \left(1-\frac{20}{120-60}\right)^{\alpha} = \left(\frac{2}{3}\right)^{\alpha}$, so to get ${}_{20}p_{60} = \frac{1}{2}$, we need $\alpha = \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{2}{3}\right)} = 1.70951129135$.

4. *An insurance company uses a survival model given by*

$$S_0(x) = \frac{1}{3}\left(1-\frac{x}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1-\frac{x}{120}\right)^{\frac{1}{3}}$$

   Using this model, prepare a life table for the ages from 40 to 45, using radix 10,000.

   We calculate

$$S_0(40) = \frac{1}{3}\left(1 - \frac{40}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1 - \frac{40}{120}\right)^{\frac{1}{3}} = 0.8780586774$$
$$S_0(41) = \frac{1}{3}\left(1 - \frac{41}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1 - \frac{41}{120}\right)^{\frac{1}{3}} = 0.874478064187$$
$$S_0(42) = \frac{1}{3}\left(1 - \frac{42}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1 - \frac{42}{120}\right)^{\frac{1}{3}} = 0.870863315825$$
$$S_0(43) = \frac{1}{3}\left(1 - \frac{43}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1 - \frac{43}{120}\right)^{\frac{1}{3}} = 0.86721361231$$
$$S_0(44) = \frac{1}{3}\left(1 - \frac{44}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1 - \frac{44}{120}\right)^{\frac{1}{3}} = 0.863528101405$$
$$S_0(45) = \frac{1}{3}\left(1 - \frac{45}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1 - \frac{45}{120}\right)^{\frac{1}{3}} = 0.859805896856$$
$$S_0(46) = \frac{1}{3}\left(1 - \frac{46}{105}\right)^{\frac{1}{4}} + \frac{2}{3}\left(1 - \frac{46}{120}\right)^{\frac{1}{3}} = 0.856046076464$$

This gives

$$p_{40} = \frac{0.874478064187}{0.8780586774} = 0.99592212536$$
$${}_2p_{40} = \frac{0.870863315825}{0.8780586774} = 0.991805375016$$
$${}_3p_{40} = \frac{0.86721361231}{0.8780586774} = 0.987648815086$$
$${}_4p_{40} = \frac{0.863528101405}{0.8780586774} = 0.983451474977$$
$${}_5p_{40} = \frac{0.859805896856}{0.8780586774} = 0.979212345355$$
$${}_6p_{40} = \frac{0.856046076464}{0.8780586774} = 0.97493037595$$

This gives us the following lifetable:

x	l_x	d_x
45	10000.00	40.78
46	9959.22	41.17
47	9918.05	41.57
48	9876.49	41.97
49	9834.51	42.39
50	9792.12	42.82

5. *Using the lifetable:*

x	l_x	d_x
35	*10000.00*	*3.91*
36	*9996.09*	*4.37*
37	*9991.72*	*4.91*
38	*9986.81*	*5.52*
39	*9981.30*	*6.21*
40	*9975.09*	*7.00*

calculate the probability that an individual aged 36 years and five months survives another 3 years, using:

(a) the uniform distribution of deaths assumption.

Under UDD, we get

$$l_{36\frac{5}{12}} = \frac{5}{12}l_{37} + \frac{7}{12}l_{36} = \frac{5}{12} \times 9991.72 + \frac{7}{12} \times 9996.09 = 9994.26916667$$

and

$$l_{39\frac{5}{12}} = \frac{5}{12}l_{40} + \frac{7}{12}l_{39} = \frac{5}{12} \times 9975.09 + \frac{7}{12} \times 9981.30 = 9978.7125$$

The probability of surviving is therefore ${}_3p_{36\frac{5}{12}} = \frac{9978.7125}{9994.26916667} = 0.998443441295.$

(b) the constant force of mortality assumption.

Under constant force of mortality, we have

$$l_{36\frac{5}{12}} = l_{37}{}^{\frac{5}{12}} l_{36}{}^{\frac{7}{12}} = 9991.72^{\frac{5}{12}} \times 9996.09^{\frac{7}{12}} = 9994.26893444$$

and

$$l_{39\frac{5}{12}} = l_{40}{}^{\frac{5}{12}} \times l_{39}{}^{\frac{7}{12}} = 9975.09^{\frac{5}{12}} \times 9981.30^{\frac{7}{12}} = 9978.71203032$$

The probability of surviving is therefore ${}_3p_{36\frac{5}{12}} = \frac{9978.71203032}{9994.26893444} = 0.9984434175.$

Standard Questions

6. *An insurance company wants to use a model of mortality of the form* $\mu_x = \frac{a}{120-x} + \frac{1}{m-x}$ *for* $x < 120$. *The company wants to ensure that the life expectancy for an individual aged 65 is 15 years and that the force of mortality at age 65 is* $\mu_{65} = \frac{1}{44}$. *What values of* a *and* b *should they use to match these values.*

We have that

$$
\begin{aligned}
{}_tp_{65} &= e^{-\int_0^t \mu_{65+s}\,ds} \\
&= e^{-\int_0^t \frac{a}{55-s}+\frac{1}{m-65-s}\,ds} \\
&= e^{[a\log(55-s)+\log(m-65-s)]_0^t} \\
&= \left(\frac{55-t}{55}\right)^a \left(\frac{m-65-t}{m-65}\right)
\end{aligned}
$$

Now this gives the life expectancy is

$$
\begin{aligned}
e_{65} &= \int_0^{55} \left(1-\frac{t}{55}\right)^a \left(1-\frac{t}{m-65}\right)\,dt \\
&= 55\int_0^1 u^a \left(1-\frac{55(1-u)}{m-65}\right)\,du \\
&= 55\left(\left(1-\frac{55}{m-65}\right)\frac{1}{a}+\frac{55}{(m-65)(a+1)}\right)
\end{aligned}
$$

We therefore need to solve

$$
\begin{aligned}
\frac{a}{55} + \frac{1}{m-65} &= \frac{1}{44} \\
\frac{1}{m-65} &= \frac{1}{44} - \frac{a}{55} \\
55\left(\left(1 - 55\left(\frac{1}{44} - \frac{a}{55}\right)\right)\frac{1}{a} + \frac{55}{(a+1)}\left(\frac{1}{44} - \frac{a}{55}\right)\right) &= 15 \\
55\left(\frac{1-\frac{55}{44}}{a} + 1 + \frac{55}{(a+1)}\left(\frac{1}{44} + \frac{1}{55} - \frac{a+1}{55}\right)\right) &= 15 \\
55\left(\frac{1-\frac{55}{44}}{a} + \frac{55}{(a+1)}\left(\frac{1}{44} + \frac{1}{55}\right)\right) &= 15 \\
55\left((1 - \frac{55}{44})(a+1) + 55\left(\frac{1}{44} + \frac{1}{55}\right)a\right) &= 15a(a+1) \\
15a^2 + \left(15 - 55(1 - \frac{55}{44}) - 55^2\left(\frac{1}{44} + \frac{1}{55}\right)\right)a - 55(1 - \frac{55}{44}) &= 0 \\
15a^2 + (15 - 55 - 55)\,a - 55(1 - \frac{55}{44}) &= 0 \\
3a^2 - 19a - 11(1 - \frac{5}{4}) &= 0 \\
a &= \frac{19 \pm \sqrt{19^2 - 33}}{6} \\
a &= 0.14820495395 \\
\frac{1}{m-65} &= \frac{1}{44} - \frac{a}{55} \\
\frac{1}{m-65} &= \frac{1}{44} - \frac{0.14820495395}{55} \qquad = 0.020032637200 \\
m - 65 &= \frac{1}{0.0200326372009} \\
&= 49.9185399292 \\
m &= 114.9185
\end{aligned}
$$

7. *An insurance company prepares the following lifetable for an individual.*

x	l_x	d_x
40	*10000.00*	*51.16*
41	*9948.84*	*59.96*
42	*9888.87*	*70.24*
43	*9818.64*	*82.19*
44	*9736.44*	*96.08*
45	*9640.36*	*112.16*
46	*9528.20*	*130.72*
47	*9397.48*	*152.04*
48	*9245.44*	*176.41*
49	*9069.03*	*204.11*
50	*8864.92*	*235.34*

Prepare a new life table for this individual over the age range 45–50 using radix 10,000.

From the lifetable we calculate

$$
\begin{aligned}
p_{45} &= \frac{9528.20}{9640.36} = 0.988365579709 \\
{}_2p_{45} &= \frac{9397.48}{9640.36} = 0.974805920111 \\
{}_3p_{45} &= \frac{9245.44}{9640.36} = 0.959034724844 \\
{}_4p_{45} &= \frac{9069.03}{9640.36} = 0.940735615682 \\
{}_5p_{45} &= \frac{8864.92}{9640.36} = 0.919563169840 \\
{}_6p_{45} &= \frac{8629.58}{9640.36} = 0.895151218419
\end{aligned}
$$

The new lifetable is therefore

x	l_x	d_x
45	10000.00	11.63
46	9883.66	13.56
47	9748.06	15.77
48	9590.35	18.30
49	9407.36	21.17
50	9195.63	24.41