ACSC/STAT 4703, Actuarial Models II
Winter 2020
Toby Kenney
Sample Final Examination
Model Solutions

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *Automobile insurance company A estimates that the standard deviation of aggregate annual claims for an individual is \$3,579 and the mean is \$1,824.*

   (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.02$, $p = 0.90$.) [5 mins.]

   The individual or group is assigned full credibility if the probability the the relative error in their mean claim as an estimate for their expected claim is less than r is at least p. For an individual with n years of history, the variance of the mean aggregate claims is $\frac{3579^2}{n}$, so the probability that the relative error is less than r is $2\Phi\left(\frac{1824\sqrt{n}}{3579r}\right) - 1$. Setting this to 0.9 means

$$\begin{aligned}
2\Phi\left(\frac{1824r\sqrt{n}}{3579}\right) - 1 &= 0.90\\
\Phi\left(\frac{1824r\sqrt{n}}{3579}\right) &= 0.95\\
\frac{1824 \times 0.02\sqrt{n}}{3579} &= 1.644854\\
n &= \left(\frac{1.644854 \times 3579}{1824 \times 0.02}\right)^2\\
&= 26041.6470236
\end{aligned}$$

   (b) If an individual has claimed \$6,000 in the past 5 years, what credibility premium should they pay? [5 mins.]

   The credibility of 5 years of experience is $Z = \sqrt{\frac{5}{26041.6470236}} = 0.0138564116865$, then the premium is

$$0.0138564116865 \times 1200 + 0.986143588314 \times 1824 = \$1815.35$$

   (c) Insurance company B uses $r = 0.10$ and $p = 0.95$ for setting its credibility premiums. The individual from (b) claims \$1,200 every year. She switches to company B, where she has 0 years of experience. How many years will it take until the total premiums she has paid is lower than if she had not switched.

The standard for full credibility for company B is the solution to

$$\begin{aligned}
2\Phi\left(\frac{1824 \times 0.1\sqrt{n}}{3579}\right) - 1 &= 0.95 \\
\Phi\left(\frac{182.4\sqrt{n}}{3579}\right) &= 0.975 \\
\frac{182.4\sqrt{n}}{3579} &= 1.959964 \\
n &= \left(\frac{1.959964 \times 3579}{182.4}\right)^2 \\
&= 1479.00593819
\end{aligned}$$

The premium after n years at company B is therefore

$$1824 - \sqrt{\frac{n}{1479.00593819}}(1824 - 1200)$$

while the premium after $n + 5$ years at company A is

$$1824 - \sqrt{\frac{n+5}{26041.6470236}}(1824 - 1200)$$

Thus the total premium paid after N years is

$$\sum_{n=0}^{N} 1824 - \sqrt{\frac{n}{1479.00593819}}(1824 - 1200)$$

for company B and

$$\sum_{n=0}^{N} 1824 - \sqrt{\frac{n+5}{26041.6470236}}(1824 - 1200)$$

for For company A. We therefore want to find when

$$\sum_{n=0}^{N} 1824 - \sqrt{\frac{n}{1479.00593819}}(1824 - 1200) < \sum_{n=0}^{N} 1824 - \sqrt{\frac{n+5}{26041.6470236}}(1824 - 1200)$$

It is straightforward to see that this is equivalent to

$$\sum_{n=0}^{N} \sqrt{\frac{n}{1479.00593819}} > \sum_{n=0}^{N} \sqrt{\frac{n+5}{26041.6470236}}$$

Multiplying by $\sqrt{1479.00593819}$ gives

$$\sum_{n=0}^{N} \sqrt{n} > \sum_{n=0}^{N} \sqrt{\frac{n+5}{17.6075337841}}$$

We compute these values:

N	$\sum_{n=0}^{N} \sqrt{n}$	$\sum_{n=0}^{N} \sqrt{\frac{n+5}{17.6075337841}}$
0	0.000000	0.5328878
1	1.000000	1.1166370
2	2.414214	1.7471583
3	4.146264	2.4212139
4	6.146264	3.1361579
5	8.382332	3.8897750

We see that after 2 years, they will have paid less in total premiums to company B.

2. *A home insurance company classifies policyholders into "low risk" and "high risk". Annual claims from low risk policyholders follow a Pareto distribution with $\theta = 1000$ and $\alpha = 5$ and claims from high risk policyholders follow a gamma distribution with $\alpha = 3$ and $\theta = 300$. 40% of policyholders are high risk.*

(a) What is the Bayesian premium for a policyholder who has claimed \$500 in one year?

The likelihood of \$500 of aggregate claims in a year is $\frac{\alpha\theta^\alpha}{(\theta+500)^{\alpha+1}} = \frac{5\times1000^5}{1500^6} = 0.000438957475995$ for a low-risk policyholder, and $\frac{500^2 e^{-\frac{500}{300}}}{300^3\Gamma(3)} = 0.000874424087213$. Therefore the posterior probability that such an individual is low risk is

$$\frac{0.6 \times 0.000438957475995}{0.6 \times 0.000438957475995 + 0.4 \times 0.000874424087213} = 0.429547437216$$

The Bayesian premium is therefore

$$0.429547437216 \times 250 + 0.570452562784 \times 900 = 620.79416581$$

(b) What is the largest Bayes premium a policyholder could have to pay in Year 2?

If the likelihood of the first year's aggregate losses is a under the Pareto distribution and b under the Gamma distribution, then the posterior probability that the individual is high risk is $\frac{0.4b}{0.4b+0.6a} = \frac{1}{1+1.5\frac{a}{b}}$, which is a decreasing function of $\frac{a}{b}$, so the Bayes premium in Year 2 is an increasing function of $\frac{b}{a}$. The largest premium in Year 2 therefore corresponds to the largest value of $\frac{b}{a}$. If x is the aggregate loss in Year 1, then we have

$$\frac{b}{a} = \frac{\left(\frac{x^2 e^{-\frac{x}{300}}}{300^3\Gamma(2)}\right)}{\left(\frac{5\times1000^5}{(1000+x)^6}\right)} = \frac{x^2(x+1000)^6 e^{-\frac{x}{300}}}{1.8\times10^{22}}$$

We find the maximum of this by setting its derivative to zero:

$$\begin{aligned}\left(2x(x+1000)^6 + 6x^2(x+1000)^5 - \frac{x^2(x+1000)^6}{300}\right)e^{-\frac{x}{300}} &= 0\\ 600(x+1000) + 1800x - x(x+1000) &= 0\\ x^2 - 1400x - 600000 &= 0\\ x &= 700 + \sqrt{700^2 + 600000}\\ &= 1744.03065089\end{aligned}$$

For this value of x, we have

$$\frac{b}{a} = \frac{1744.03065089^2 \times 2744.03065089^6 e^{-\frac{1744.03065089}{300}}}{1.8 \times 10^{22}} = 215.489696985$$

Thus the posterior probability that the individual is high-risk is

$$\frac{215.489696985}{215.489696985 + 1} = 0.995380842535$$

and the Bayes premium is

$$0.995380842535 \times 900 + 0.004619157465 \times 250 = \$897.00$$

3. *A policyholder starts a new auto insurance policy. In the first year, he pays the book premium of \$760, and his aggregate claims are \$850. His premium for the second year is \$774, while the book premium is still \$760. This premium is calculated using Buhlmann credibility. If he claims \$420 in the second year, what premium will he pay in the third year (assuming the book premium remains \$760).*

Under the Bühlmann credibility model, the credibility of n years of experience is $\frac{n}{n+K}$ where $K = \frac{\text{EPV}}{\text{VHM}}$. From the individual's Year 2 premium, we have

$$\begin{aligned} 850Z + 760(1 - Z) &= 774 \\ Z &= \frac{14}{90} \\ \frac{1}{1+K} &= \frac{14}{90} \\ K &= \frac{76}{14} = \frac{38}{7} \end{aligned}$$

Therefore, the credibility of two years of experience is

$$\frac{2}{2 + \frac{38}{7}} = \frac{14}{52} = \frac{7}{26}$$

The individual's premium in Year 3 is therefore

$$\frac{7}{26} \times 635 + \frac{19}{26} \times 760 = \$726.35$$

4. *A health insurance company is reviewing the premium for a group with the following past claim history:*

Year	*1*	*2*	*3*	*4*	*5*
Exposure	*352*	*532*	*235*	*364*	*403*
Aggregate claims	*\$35,633*	*\$42,014*	*\$26,852*	*\$63,154*	*\$33,706*

The book premium per unit of exposure is \$153.59. The expected process variance is 23145 and the variance of hypothetical means is 303 (both per unit of exposure).

(a) Calculate the premium per unit of exposure in Year 6.

The credibility of 1886 units of exposure is $\frac{1886}{1886+\frac{23145}{303}} = 0.961074868442$. Therefore the premium per unit of exposure is $0.961074868442 \times \frac{201359}{1886} + 0.038925131558 \times 153.59 = \$108.587776298$

(b) If the company has 490 units of exposure in Year 6, what aggregate claims in Year 6 would cause it to have the same premium per unit of exposure in Year 7?

If the company has 490 units of exposure in Year 6, then the credibility of its 6 years of experience is $\frac{2376}{2376+\frac{23145}{303}} = 0.968852320028$. In order for the premium to be 108.587776298 per unit of exposure, we need its aggregate claims X in Year 6 to satisfy:

$$0.968852320028\frac{201359 + X}{2376} + 0.031147679972 \times 153.59 = 108.587776298$$
$$X = (108.587776298 - 0.031147679972 \times 153.59) \times \frac{2376}{0.968852320028} - 201359$$
$$= 53208.010386$$

5. *An insurance company has 3 years of past history on a homeowner, denoted X_1, X_2, X_3. Because the individual moved house at the end of the second year, the third year has a different mean and variance, and is not as correlated with the other two years. It has the following*

$$\begin{aligned}
\mathbb{E}(X_1) &= 1,322 & \mathrm{Var}(X_1) &= 226,000\\
\mathbb{E}(X_2) &= 1,322 & \mathrm{Var}(X_2) &= 226,000\\
\mathbb{E}(X_3) &= 4,081 & \mathrm{Var}(X_3) &= 1,108,000\\
\mathbb{E}(X_4) &= 4,081 & \mathrm{Var}(X_4) &= 1,108,000\\
\mathrm{Cov}(X_1, X_2) &= 214 & \mathrm{Cov}(X_1, X_3) &= 181\\
\mathrm{Cov}(X_2, X_3) &= 181 & \mathrm{Cov}(X_1, X_4) &= 181\\
\mathrm{Cov}(X_2, X_4) &= 181 & \mathrm{Cov}(X_3, X_4) &= 861
\end{aligned}$$

It uses a formula $\hat{X}_4 = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$ to calculate the credibility premium in the fourth year. Calculate the values of α_0, α_1, α_2 and α_3. [15 mins.]

The company needs to choose α_0, α_1, α_2 and α_3 to satisfy:

$$\begin{aligned}
\mathbb{E}(X_4) &= \alpha_0 + \alpha_1\mathbb{E}(X_1) + \alpha_2\mathbb{E}(X_2) + \alpha_3\mathbb{E}(X_3)\\
\mathrm{Cov}(X_4, X_1) &= \alpha_1 \mathrm{Var}(X_1) + \alpha_2 \mathrm{Cov}(X_2, X_1) + \alpha_3 \mathrm{Cov}(X_3, X_1)\\
\mathrm{Cov}(X_4, X_2) &= \alpha_1 \mathrm{Cov}(X_1, X_2) + \alpha_2 \mathrm{Var}(X_2) + \alpha_3 \mathrm{Cov}(X_3, X_2)\\
\mathrm{Cov}(X_4, X_1) &= \alpha_1 \mathrm{Cov}(X_1, X_3) + \alpha_2 \mathrm{Cov}(X_2, X_3) + \alpha_3 \mathrm{Var}(X_3)
\end{aligned}$$

Substituting the values gives:

$$\begin{aligned}4081 &= \alpha_0 + 1322\alpha_1 + 1322\alpha_2 + 4081\alpha_3\\ 181 &= 226000\alpha_1 + 214\alpha_2 + 181\alpha_3\\ 181 &= 214\alpha_1 + 226000\alpha_2 + 181\alpha_3\\ 861 &= 181\alpha_1 + 181\alpha_2 + 1108000\alpha_3\end{aligned}$$

By symmetry, we see that α_1 and α_2 are equal. This gives

$$\begin{aligned}181 &= 226214\alpha_1 + 181\alpha_3\\ 861 &= 362\alpha_1 + 1108000\alpha_3\\ 226214 \times 861 - 362 \times 181 &= (226214 \times 1108000 + 362 \times 181)\alpha_3\\ \alpha_3 &= \frac{194704732}{250,645,046,478} = 0.0007768146\\ \alpha_1 &= \frac{181 - 181 \times 0.0007768146}{226214} = 0.0007995058\\ \alpha_0 &= 4081 - 1322 \times 2 \times 0.0007995058 - 4081 \times 0.0007768146 = 4075.716\end{aligned}$$

The values are:

$$\begin{aligned}\alpha_0 &= 4075.716\\ \alpha_1 &= 0.0007995058\\ \alpha_2 &= 0.0007995058\\ \alpha_3 &= 0.0007768146\end{aligned}$$

6. *An insurance company has the following previous data on aggregate claims:*

Policyholder	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Mean*	*Variance*
1	*1,210*	*246*	*459*	*1,461*	*944.00*	*340158.00*
2	*0*	*0*	*0*	*0*	*0.00*	*0.00*
3	*0*	*2,185*	*0*	*0*	*548.25*	*1202312.25*
4	*809*	*0*	*0*	*1,725*	*633.50*	*674939.00*
5	*0*	*0*	*0*	*0*	*0.00*	*0.00*

Calculate the Bühlmann credibility premium for policyholder 3 in Year 5. [15 mins.]

The expected process variance is $\frac{1}{5}(340158 + 0 + 1202312.25 + 674939 + 0) = 443421.85$. The population mean is $\frac{944+0+548.25+633.50+0}{5} = 405.15$.

total variance of estimated means is $\frac{(944-405.15)^2+(-405.15)^2+(548.25-405.15)^2+(633.50-405.15)^2+(-405.15)^2}{4} = 172318.425$. The variance of hypothetical means is therefore $172318.425 - \frac{443421.85}{4} = 61462.96$. The credibility of 4 years of experience is therefore $\frac{4}{4+\frac{443421.85}{61462.96}} = 0.3566825$. The premium for policyholder 3 is therefore $0.3566825 \times 548.25 + 0.6433175 \times 405.15 = \$456.19$.

7. *An insurance company collects the following claim frequency data for 7,000 customers insured for the past 3 years:*

No. of claims	*Frequency*
0	*1,492*
1	*2,460*
2	*1,810*
3	*827*
4	*302*
5	*72*
6	*31*
7	*3*
8	*1*
>8	*0*

It assumes that the number of claims an individual makes in a year follows a Poisson distribution with parameter Λ*, which may vary between individuals.*

Find the credibility estimate for the expected number of claims per year for an individual who has made 4 claims in the past 3 years. [15 mins.]

The total number of claims in the past 3 years was $1\times2460+2\times1810+3\times827+4\times302+5\times72+6\times31+7\times3+8\times1 = 10,344$. The total number of policyholders is $1491+2461+1810+831+302+72+30+2+1 = 7,000$. The average number of claims per policyholder per year is therefore $\frac{10344}{21000} = 0.492571428571$. This is also the expected process variance. The variance of estimated means is

$$\frac{1}{6999}\left(1493\times0.492571428571^2 + 2460\left(\frac{1}{3}-0.492571428571\right)^2 + 1810\left(\frac{2}{3}-0.492571428571\right)^2\right.$$
$$+827(1-0.492571428571) + 307\left(\frac{4}{3}-0.492571428571\right)^2 + 72\left(\frac{5}{3}-0.492571428571\right)^2$$
$$\left.+31(2-0.492571428571)^2 + 3\left(\frac{7}{3}-0.492571428571\right)^2 + \left(\frac{8}{3}-0.492571428571\right)^2\right) = 0.185843829802$$

The variance due to the Poisson sampling is $\frac{0.492571428571}{3} = 0.16419047619$. Therefore, the variance of hypothetical means is $0.185843829802-0.16419047619 = 0.021653353612$. The credibility of 3 year's experience is $\frac{3}{3+\frac{0.492571428571}{0.021653353612}} = 0.116513707423$. The expected number of claims is therefore $0.116513707423\times\frac{4}{3}+0.883486292577\times0.492571428571 = 0.590531715155$.

8. *An actuary is reviewing claim data from accident year 2019 for a particular line of insurance. The earned premium is \$3,520,320, and the aggregate claims are \$2,560,600. At the start of the year, there are 7,400 policies in force. After 4 months, there are only 4,200 policies in force. After 8 months, there are 7,600 policies in force, and at the end of the year, there are 8,500 policies in force. Assuming the number of policies in force is linear between these data points, what should be the percentage change in the premium for policy year 2021, if inflation is 5% and the permissible loss ratio is 0.80? Assume that policies are sold uniformly during 2021.*

The loss ratio is $\frac{2560600}{3520320} = 0.727377056631$, so without inflation, the premium would be adjusted by a factor $\frac{0.727377056631}{0.8} = 0.909221320789$. The average number of policies in force in 2019 is

$$\frac{1}{12}\left(4\times\frac{7400+4200}{2}+4\times\frac{4200+7600}{2}+4\times\frac{7600+8500}{2}\right)=6583.33333333$$

We calculate the average inflation from the start of 2019 as the following integral:

$$\begin{aligned}
&\int_0^{\frac{1}{3}}\left(\frac{7400}{6583.33333333}-\frac{9600}{6583.33333333}t\right)(1.05)^t\,dt+\int_0^{\frac{1}{3}}\left(\frac{4200}{6583.33333333}+\frac{10200}{6583.33333333}t\right)(1.05)^{t+\frac{1}{3}}\,dt\\
&\qquad+\int_0^{\frac{1}{3}}\left(\frac{7600}{6583.33333333}+\frac{2700}{6583.33333333}t\right)(1.05)^{t+\frac{2}{3}}\,dt\\
=&\frac{7400+4200(1.05)^{\frac{1}{3}}+7600(1.05)^{\frac{2}{3}}}{6583.33333333}\int_0^{\frac{1}{3}}(1.05)^t\,dt+\frac{10200(1.05)^{\frac{1}{3}}+2700(1.05)^{\frac{2}{3}}-9600}{6583.33333333}\int_0^{\frac{1}{3}}t(1.05)^t\,dt\\
=&2.96508341927\left[\frac{1.05^t}{\log(1.05)}\right]_0^{\frac{1}{3}}+0.540229220628\left(\left[t\frac{(1.05)^t}{\log(1.05)}\right]_0^{\frac{1}{3}}-\int_0^{\frac{1}{3}}\frac{(1.05)^t}{\log(1.05)}\,dt\right)\\
=&\frac{2.96508341927}{\log(1.05)}\left(1.05^{\frac{1}{3}}-1\right)+\frac{0.540229220628}{\log(1.05)}\left(\frac{(1.05)^{\frac{1}{3}}}{3}+\frac{1-(1.05)^{\frac{1}{3}}}{\log(1.05)}\right)\\
=&1.02678207043
\end{aligned}$$

For a policy sold in 2021, the average inflation from the start of 2021 to the accident time is:

$$\begin{aligned}
\int_0^1 t(1.05)^t\,dt+\int_1^2(2-t)(1.05)^t\,dt &= \int_0^1 t(1.05)^t\,dt+1.05\int_0^1(1-t)(1.05)^t\\
&=1.05\int_0^1(1.05)^t-0.05\int_0^1 t(1.05)^t\,dt\\
&=1.05\int_0^1(1.05)^t-0.05\left[t\frac{(1.05)^t}{\log(1.05)}\right]_0^1+\frac{0.05}{\log(1.05)}\int_0^1(1.05)^t\,dt\\
&=\left(1.05+\frac{0.05}{\log(1.05)}\right)\frac{0.05}{\log(1.05)}-0.05\times\frac{1.05}{\log(1.05)}\\
&=\left(\frac{0.05}{\log(1.05)}\right)^2\\
&=1.05020830855
\end{aligned}$$

Therefore, the premium for 2021 needs to be adjusted by a factor $\frac{0.909221320789\times1.05^2\times1.05020830855}{1.02678207043} = 1.02528683907$ so the percentage increase is 2.5287%.

9. *An insurer classifies policies into three classes — low risk, medium risk, and high risk. The experience from policy year 2018 is:*

Class	*Current differential*	*Earned premiums*	*Loss payments*
low risk	*0.72*	*4,740*	*3,940*
medium risk	*1*	*4,490*	*3,880*
high risk	*1.68*	*5,670*	*4,930*

The base premium was \$420. Claim amounts are subject to 5% annual inflation. If the expense ratio is 30%:

(a) calculate the new premiums for each age class for policy year 2021. [15 mins]

The new differentials are $0.72\times\frac{3940\times4490}{3880\times4740}=0.6925720997$ for low risk and $1.68\times\frac{4930\times4490}{3880\times5670}=1.69039327988$ for high risk. Balancing back to these differentials, the earned premiums would be $4740\times\frac{3940\times4490}{3880\times4740}=4559.43298969$ for low risk and $5670\times\frac{4930\times4490}{3880\times5670}=5705.0773196$ for high risk, so the total earned premiums would be $5705.0773196+4490+4559.43298969=\$14754.5103093$. The total losses were $3940+3880+4930=12750$, so without inflation, the base premium needs to be adjusted by a factor $\frac{12750}{0.7\times14754.5103093}=1.23448934139$. From the start of 2018 to an accident time in 2018, the average inflation is $\int_0^1(1.05)^t\,dt=\frac{0.05}{\log(1.05)}=1.02479671572$. From the start of 2021 to a random accident time from policy year 2021, inflation is

$$\begin{aligned}
\int_0^1 t(1.05)^t\,dt+\int_1^2(2-t)(1.05)^t\,dt &= \int_0^1 t(1.05)^t\,dt+1.05\int_0^1(1-t)(1.05)^t\\
&=1.05\int_0^1(1.05)^t-0.05\int_0^1 t(1.05)^t\,dt\\
&=1.05\int_0^1(1.05)^t-0.05\left[t\frac{(1.05)^t}{\log(1.05)}\right]_0^1+\frac{0.05}{\log(1.05)}\int_0^1(1.05)^t\,dt\\
&=\left(1.05+\frac{0.05}{\log(1.05)}\right)\frac{0.05}{\log(1.05)}-0.05\times\frac{1.05}{\log(1.05)}\\
&=\left(\frac{0.05}{\log(1.05)}\right)^2\\
&=1.05020830855
\end{aligned}$$

Thus the base premium should be adjusted by a factor $\frac{1.23448934139\times1.05020830855\times(1.05)^3}{1.02479671572}=1.4645121083$, so the new base premium is $1.4645121083\times420=615.095085486$ The premiums are:

low risk	$615.095085486\times0.6925720997=\$426.00$
high risk	$615.095085486\times1.69039327988=\$1,039.75$

(b) The insurance company wants to reduce to two policy classes, with low risk as the base class. It will assign some policyholders from the medium risk class into low risk, and some into high risk. Assume that the loss ratio for the medium-risk policyholders reclassified as low-risk and the loss ratio for the medium-risk policyholders reclassified as high-risk are the same. If the new base premium after combining these classes is \$480, what should the differential for the high risk class be?

We have that the original numbers of policyholders in each class are

Risk Class	Number of policyholders
Low	$\frac{4740}{0.72\times420}=15.6746031746$
Medium	$\frac{4490}{420}=10.6904761905$
High	$\frac{5670}{1.68\times420}=8.03571428571$

The total new premiums should be equal to the original total premiums. From (a), we have that these are $15.6746031746\times615.095085486\times0.6925720997+10.6904761905\times615.095085486+8.03571428571\times615.095085486\times1.69039327988=21608.159$

Let the proportion of medium-risk individuals reclassified as high-risk be p. Let the new differential be d, then the new total premiums are

$$((15.6746031746+10.6904761905(1-p))+(10.6904761905p+8.03571428571)d)480=21608.159$$

If we adjust the premiums for the new classification, then the loss ratio for low-risk in 2018 with the differential set to 1 is $\frac{3940+3880(1-p)}{420(15.6746031746+10.6904761905(1-p))}$, and the loss ratio for high-risk in 2018 with differential set to 1 is $\frac{4930+3880p}{420(8.03571428571+10.6904761905p)}$. The new differential is therefore

$$d = \frac{4930+3880p}{420(8.03571428571+10.6904761905p)} \times \frac{420(15.6746031746+10.6904761905(1-p))}{3940+3880(1-p)} = \frac{4930+3880p}{8.03571428571+10.6904761905p}$$

We therefore want to solve:

$$\begin{aligned}
\frac{4930+3880p}{8.03571+10.6905p} \times \frac{15.6746+10.6905(1-p)}{3940+3880(1-p)} &= d \\
((15.6746+10.6905(1-p))+(10.6905+8.03571)d)480 &= 21608.159 \\
\frac{4930+3880p}{8.03571+10.6905p} \times \frac{15.6746+10.6905(1-p)}{3940+3880(1-p)} &= \frac{45.0169979167-(15.6746+10.6905(1-p))}{10.6905p+8.03571} \\
(4930+3880p)(15.6746+10.6905(1-p)) &= (18.6519+10.6905p)(3940+3880(1-p)) \\
289374559p^2+97744242p-90028678 &= 0 \\
p &= \frac{-97744242+\sqrt{97744242^2+4\times 289374559\times 90028678}}{2\times 289374559} \\
&= 0.413896002018
\end{aligned}$$

This gives

$$d = \frac{18.6518979167+10.6904761905\times 0.413896002018}{10.6904761905\times 0.413896002018+8.03571428571} = 1.85198972889$$

10. *An insurer has different premiums for personal and commercial vehicles. Its experience for accident year 2018 is given below. There was a rate change on 1st August 2017, which affects some policies in 2018.*

Type	*Differential before rate change*	*Current differential*	*Earned premiums*	*Loss payments*
Personal	*1*	*1*	*11,300*	*9,800*
Commercial	*1.51*	*1.67*	*7,600*	*6,300*

Before the rate change, the base premium was \$950. The current base premium is \$1,020.

(a) Assuming that policies were sold uniformly over the year, calculate the new premimums for policy year 2020 assuming 6% annual inflation and a permissible loss ratio of 0.75. [15 mins]

The old premium applied for $\frac{7}{12}$ of 2017. Policies with this premium were therefore in force for $\frac{1}{2}\left(\frac{7}{12}\right)^2 = \frac{49}{288}$ of earned premium in 2018. Adjusting to the new premiums, the earned premium for personal in 2018 is $11300 \times \frac{1020}{1020\times\frac{239}{288}+950\times\frac{49}{288}} = 11433.4998106$. The adjusted earned premium for commercial policies in 2018 is $7600 \times \frac{1.67\times 1020}{1.67\times 1020\times\frac{239}{288}+1.51\times 950\times\frac{49}{288}} = 7809.75640923$.

This means that the adjusted loss ratios are $\frac{9800}{11433.4998106} = 0.85713037673$ and $\frac{6300}{7809.75640923} = 0.80668329073$. The differential needs to be adjusted by a factor of $\frac{0.80668329073}{0.85713037673}$, so the new differential is $1.67 \times \frac{0.80668329073}{0.85713037673} = 1.57171082964$. Using this differential, total adjusted earned premiums in 2016 would be $11433.4998106+7809.75640923\times\frac{0.80668329073}{0.85713037673} = 18783.6068317$. The loss ratio is then $\frac{16100}{18783.6068317} = 0.85713037673$. The target loss ratio is 0.75, so without inflation, premiums need to be increased by a factor $\frac{0.85713037673}{0.75} = 1.14284050231$. Losses in accident year

2018 experience average inflation $\int_0^1 e^{\log(1.06)t}\,dt = \frac{0.06}{\log(1.06)} = 1.02970867194$ from the start of the year, while losses in policy year 2020 experience average inflation

$$\begin{aligned}\int_0^1 te^{\log(1.06)t}\,dt + 1.06\int_0^1 (1-t)e^{\log(1.06)t}\,dt &= 1.06\int_0^1 e^{\log(1.06)t}\,dt - 0.06\int_0^1 te^{\log(1.06)t}\,dt\\ &= 1.06 \times \frac{0.06}{\log(1.06)} - 0.06\left(\frac{1.06}{\log(1.06)} - \frac{0.06}{\log(1.06)^2}\right)\\ &= 1.06029994908\end{aligned}$$

from the start of 2020. The base premium therefore needs to change by a factor $1.14284050231 \times (1.06)^2 \times \frac{1.06029994908}{1.02970867194} = 1.32224436298$. The new base premium is $1.32224436298 \times 1020 = \$1,348.69$, and the new premium for commercial policies is $1348.68925024 \times 1.57171082964 = \$2,119.75$.

(b) Suppose that twice as many policies are sold in April, May, June, July, August, September and October as in other months. What is the new premium in this case. [This rate of sale applies to both the rate change in the data, and the policies sold in 2020.] [15 mins]

$\frac{3}{19}$ of the policies sold in 2017 are sold in January, February or March, and these policies are in force for an average of $\frac{3}{2\times 12}$ of 2018. $\frac{8}{19}$ of policies sold in 2017 are sold in April, May, June or July, and these policies are in force for an average of $\frac{5}{12}$ of 2018. Thus the proportion of earned premiums for 2018 that are under the old premiums is $\frac{3}{19} \times \frac{3}{24} + \frac{8}{19} \times \frac{5}{12} = \frac{89}{456}$. Adjusting to the new premiums, the earned premium for personal in 2018 is $11300 \times \frac{1020}{1020\times\frac{367}{456}+950\times\frac{89}{456}} = 11453.411493$. The adjusted earned premium for commercial policies in 2018 is $7600 \times \frac{1.67\times 1020}{1.67\times 1020\times\frac{367}{456}+1.51\times 950\times\frac{89}{456}} = 7841.604063$.

This means that the adjusted loss ratios are $\frac{9800}{11453.411493} = 0.855640261069$ and $\frac{6300}{7841.604063} = 0.803407051591$. The differential needs to be adjusted by a factor of $\frac{0.803407051591}{0.855640261069}$, so the new differential is $1.67 \times \frac{0.803407051591}{0.855640261069} = 1.56805358187$. Using this differential, total adjusted earned premiums in 2016 would be $11433.4998106 + 7809.75640923 \times \frac{0.803407051591}{0.855640261069} = 18766.5037094$. The loss ratio is then $\frac{16100}{18766.5037094} = 0.857911534791$. The target loss ratio is 0.75, so without inflation, premiums need to be increased by a factor $\frac{0.857911534791}{0.75} = 1.14388204639$. Because policies are sold at the same times each year, the number of policies in force is uniform over the year. Therefore, losses in accident year 2018 experience average inflation $\int_0^1 e^{\log(1.06)t}\,dt = \frac{0.06}{\log(1.06)} = 1.02970867194$ from the start of the year. Losses in policy year 2020 experience average inflation

$$\int_0^{\frac{3}{12}} \frac{12}{19}t(1.06)^t\,dt + \int_{\frac{3}{12}}^{\frac{10}{12}} \left(\frac{24}{19}t - \frac{3}{19}\right)(1.06)^t\,dt + \int_{\frac{10}{12}}^{1} \left(\frac{12}{19}t + \frac{7}{19}\right)(1.06)^t\,dt + (1.06)\int_0^{\frac{3}{12}} \left(1 - \frac{12}{19}t\right)(1.06)^t\,dt$$
$$+ (1.06)\int_{\frac{3}{12}}^{\frac{10}{12}} \left(\frac{22}{19} - \frac{24}{19}t\right)(1.06)^t\,dt + (1.06)\int_{\frac{10}{12}}^{1} \left(\frac{12}{19} - \frac{12}{19}t\right)(1.06)^t\,dt$$
$$=1.06\int_0^{\frac{3}{12}} (1.06)^t\,dt + \frac{20.32}{19}\int_{\frac{3}{12}}^{\frac{10}{12}} (1.06)^t\,dt + \frac{19.72}{19}\int_{\frac{10}{12}}^{1} (1.06)^t\,dt - \frac{0.72}{19}\int_0^{\frac{3}{12}} t(1.06)^t\,dt$$
$$- \frac{1.44}{19}\int_{\frac{3}{12}}^{\frac{10}{12}} t(1.06)^t\,dt - \frac{0.72}{19}\int_{\frac{10}{12}}^{1} t(1.06)^t\,dt$$
$$=\frac{1.06\left((1.06)^{\frac{3}{12}} - 1\right) + \frac{20.32}{19}\left((1.06)^{\frac{10}{12}} - (1.06)^{\frac{3}{12}}\right) + \frac{19.72}{19}\left((1.06) - (1.06)^{\frac{10}{12}}\right)}{\log(1.06)}$$
$$- \frac{\frac{0.72}{19} \times \frac{3}{12}(1.06)^{\frac{3}{12}} + \frac{1.44}{19}\left(\frac{10}{12}(1.06)^{\frac{10}{12}} - \frac{3}{12}(1.06)^{\frac{3}{12}}\right) + \frac{0.72}{19}\left(1.06 - \frac{10}{12}(1.06)^{\frac{10}{12}}\right)}{\log(1.06)}$$
$$+ \frac{\frac{0.72}{19}\left(1.06^{\frac{3}{12}} - 1\right) + \frac{1.44}{19}\left(1.06^{\frac{10}{12}} - 1.06^{\frac{3}{12}}\right) + \frac{0.72}{19}\left(1.06 - 1.06^{\frac{10}{12}}\right)}{\log(1.06)^2}$$
$$=1.06121297325$$

from the start of 2020. The base premium therefore needs to change by a factor $1.14388204639 \times (1.06)^2 \times \frac{1.061121297325}{1.02970867194} = 1.32458903149$. The new base premium is $1.32458903149 \times 1020 = \$1,351.08$, and the new premium for commercial policies is $1351.08081212 \times 1.56805358187 = \$2,118.57$.

11. *An insurance company has the following data for accident year 2017 when the base premium was \$840:*

		Earned Premiums		*Loss Payments*	
		House	*Apartment*	*House*	*Apartment*
Differential		*1*	*0.88*	*1*	*0.88*
Halifax	*1*	*5,200*	*4,100*	*4,350*	*?,???*
Dartmouth	*0.84*	*3,700*	*2,900*	*3,020*	*2,230*
Bedford	*1.25*	*4,400*	*2,500*	*3,550*	*2,330*

Unfortunately, some records have been lost. The base premium for policy year 2020 using this data, inflation of 3% per year and expense ratio of 0.2 was calculated at \$935. What is the missing value in the table?

Under uniform selling of policies, the inflation from the start of 2017 to the time of a random accident in 2017 is

$$\int_0^1 (1.03)^t\,dt = \frac{0.03}{\log(1.03)} = 1.01492610407$$

The inflation from the start of 2020 to a random claim in policy year 2020 is

$$\int_0^1 t(1.03)^t\,dt + \int_1^2 (2-t)(1.03)^t\,dt = 1.03\frac{0.03}{\log(1.03)} + \frac{0.03^2}{\log(1.03)^2} - 0.03\frac{1.03}{\log(1.03)} = \left(\frac{0.03}{\log(1.03)}\right)^2 = 1.03007499672$$

Therefore, the new premium without inflation is

$$\frac{935 \times 1.01492610407}{1.03007499672(1.03)^3} = 843.073646511$$

This means that the observed loss ratio is

$$\frac{843.073646511}{840} \times 0.8 = 0.802927282392$$

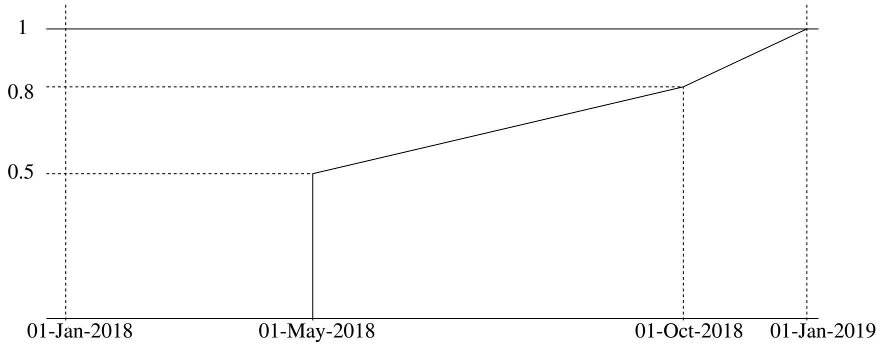
Let x be the lost value. We calculate the new differentials. For apartment the differential is $0.88 \times \frac{13300(4560+x)}{10920\times9500} = 0.88 \times 0.000128205128205(x+4560)$. For Dartmouth it is $0.84 \times \frac{5250\times9300}{6600(4350+x)} = 0.84 \times \frac{7397.72727273}{x+4350}$. For Bedford it is $1.25 \times \frac{5880\times9300}{6900(4350+x)} = 1.25 \times \frac{7925.2173913}{x+4350}$. The adjusted earned premiums are therefore

$$\begin{aligned}
5200 + 0.525641025641(x+4560) + 3700 \times \frac{7397.72727273}{x+4350} + 2750.43707307 \times \frac{x+4560}{x+4350}\\
+ 4400 \times \frac{7925.2173913}{x+4350} + 2540.13377925 \times \frac{x+4560}{x+4350}\\
=12887.4939293 + 0.525641025641x + \frac{63406473.0183}{x+4350}
\end{aligned}$$

We therefore need to solve

$$\begin{aligned}
\frac{15480+x}{12887.4939293 + 0.525641025641x + \frac{63406473.0183}{x+4350}} &= 0.802927282392\\
15480 + x &= 10347.7204775 + 0.422051520232x + \frac{50910787.0666}{x+4350}\\
(x+4350)(5132.2795225 + 0.577948479768x) &= 50910787.0666\\
0.577948479768x^2 + 7646.35540949x - 28585371.1437 &= 0\\
x &= \frac{\sqrt{7646.3554^2 + 4 \times 0.577948 \times 28585371} - 7646.3554}{2 \times 0.577948}\\
&= 3039.93566702
\end{aligned}$$

12. *An insurance company is calculating the premium for a new line of insurance it started in 2018. The new line of insurance started on 1st May 2018, and half of the policies started at that time. Due to an advertising campaign, the rate of policy purchases in November and December was twice the rate for the months from May to October. The annual premium in 2018 was \$600. The total premiums collected in 2018 were \$1,200,000 and the total losses were \$462,000. Assuming losses are uniformly distributed throughout the year, annual inflation is 5%, and the expense ratio is 0.2, calculate the new premium for policy year 2020.*



The number of policies in force at time t in the year 2018 is

$$f(t) = \begin{cases} 0 & \text{if } t < \frac{4}{12} \\ 0.6t + 0.3 & \text{if } \frac{4}{12} < t < \frac{10}{12} \\ 1.2t - 0.2 & \text{if } \frac{10}{12} < t < 1 \end{cases}$$

The total earned premiums for accident year 2018 are $1200000 \times \left(\frac{1}{2} \times \frac{6}{12} \times \left(\frac{1}{2} + 0.8\right) + \frac{1}{2} \times \frac{2}{12} \times (0.8 + 1)\right) = 570000$
The loss ratio is therefore $\frac{462000}{570000} = 0.810526315789$, so before inflation the premium is should be adjusted by a factor of $\frac{0.810526315789}{0.8} = 1.01315789474$.
Inflation from the start of 2018 to the average accident time in 2018 is given by

$$\frac{\int_{\frac{4}{12}}^{\frac{10}{12}} (0.6t + 0.3)\,(1.05)^t\,dt + \int_{\frac{10}{12}}^{1} (1.2t - 0.2)\,(1.05)^t\,dt}{\left(\frac{1}{2} \times \frac{6}{12} \times \left(\frac{1}{2} + 0.8\right) + \frac{1}{2} \times \frac{2}{12} \times (0.8 + 1)\right)}$$

$$= \frac{0.3\int_{\frac{4}{12}}^{\frac{10}{12}} (1.05)^t\,dt + 0.6\int_{\frac{4}{12}}^{\frac{10}{12}} t(1.05)^t\,dt + 1.2\int_{\frac{10}{12}}^{1} t(1.05)^t\,dt - 0.2\int_{\frac{10}{12}}^{1} (1.05)^t\,dt}{0.475}$$

$$= \frac{0.3(1.05^{\frac{10}{12}} - 1.05^{\frac{4}{12}}) - 0.2(1.05 - 1.05^{\frac{10}{12}}) + 0.6\left(\frac{10}{12}(1.05)^{\frac{10}{12}} - \frac{4}{12}(1.05)^{\frac{4}{12}} - \frac{(1.05)^{\frac{10}{12}} - (1.05)^{\frac{4}{12}}}{\log(1.05)}\right) + 1.2\left(1.05 - \frac{10}{12}1.05^{\frac{10}{12}} - \frac{1-1}{\log}\right.}{0.475\log(1.05)}$$

$$= \frac{1.05 - 0.5(1.05)^{\frac{4}{12}} - \frac{1.2(1.05) - 0.6(1.05)^{\frac{10}{12}} - 0.6(1.05)^{\frac{4}{12}}}{\log(1.05)}}{0.475\log(1.05)}$$

$= 1.03492570259$

Inflation from the start of 2020 to the average accident time in policy year 2020 is given by

$$\begin{aligned}\int_0^1 t(1.05)^t\,dt + \int_1^2 (2-t)(1.05)^t\,dt &= \int_0^1 t(1.05)^t\,dt + (1.05)\int_0^1 (1-t)(1.05)^t\,dt\\ &= 1.05\int_0^1 (1.05)^t\,dt - 0.05\int_0^1 t(1.05)^t\,dt\\ &= \frac{1.05\times 0.05}{\log(1.05)} - 0.05\left(\frac{1.05}{\log(1.05)} - \frac{0.05}{\log(1.05)^2}\right)\\ &= \left(\frac{0.05}{\log(1.05)}\right)^2\\ &= 1.05020830855\end{aligned}$$

The premium is therefore $600 \times 1.01315789474 \times 1.05^2 \times \frac{1.05020830855}{1.03492570259} = \$680.10$.

13. *An insurance company has the following data on its policies:*

Policy limit	*Losses Limited to*			
	20,000	*50,000*	*100,000*	*500,000*
20,000	*1,400,000*			
50,000	*7,540,000*	*8,010,000*		
100,000	*22,600,000*	*24,100,000*	*28,700,000*	
500,000	*5,900,000*	*6,220,000*	*6,650,000*	*6,920,000*

(a) Use this data to calculate the ILF from \$20,000 to \$500,000 using the incremental method. [5 mins]

Using the incremental method the ILFs are:

\$20,000–\$50,000	$\frac{8010000+24100000+6220000}{7540000+22600000+5900000} = 1.06354051054$
\$50,000–\$100,000	$\frac{28700000+6650000}{24100000+6220000} = 1.16589709763$
\$100,000–\$500,000	$\frac{6920000}{6650000} = 1.04060150376$

So the ILF is $1.06354051054 \times 1.16589709763 \times 1.04060150376 = 1.29032379813$.

(b) A reinsurance company uses the ILF calculated in (a) to calculate its premiums. The reinsurance company offers excess-of-loss reinsurance of \$450,000 over \$50,000 for a premium of \$240, which includes a 20% loading. How many policies in the dataset above had a policy limit of at least \$50,000? [10 mins]

The ILF from 50000–500000 is $1.16589709763 \times 1.04060150376 = 1.21323427302$ If the pure premium for policies with limit \$50,000 is x, then the expected payment on the reinsurance policy is $(1.21323427302-1)x = 0.21323427302x$. We have that $1.2 \times 0.21323427302x = 240$ so $x = 937.935525877$. The total losses limited to \$50,000 were $8010000 + 24100000 + 6220000 = \$38,330,000$. Since the average loss was 937.935525877, we have that the number of policies was $\frac{38330000}{937.935525877} = 40,866$.

14. *For a certain line of insurance, the loss amount per loss is modelled as a Pareto distribution with $\alpha = 5$. The policy has a deductible per loss set at \$5,000 and a policy limit set at \$1,000,000. After inflation of 5%, the expected payment per loss increases by 4.6%. What was the mean loss amount before the inflation? [10 mins]*

Let θ be the scale parameter of the loss distribution before inflation. The expected payment per loss is given by

$$\begin{aligned}\int_{5000}^{1000000}\left(\frac{\theta}{x+\theta}\right)^5\,dx &= \int_{5000+\theta}^{1000000+\theta}\theta^5u^{-5}\,du\\ &= \frac{\theta^5}{4}\left(\frac{1}{(5000+\theta)^4}-\frac{1}{(1000000+\theta)^4}\right)\end{aligned}$$

After inflation, θ is replaced by 1.05θ. Therefore, the expected payment per loss after inflation is

$$\frac{1.05^5\theta^5}{4}\left(\frac{1}{(5000+1.05\theta)^4}-\frac{1}{(1000000+1.05\theta)^4}\right)$$

θ is therefore the solution to the equation

$$\begin{aligned}1.046\frac{\theta^5}{4}\left(\frac{1}{(5000+\theta)^4}-\frac{1}{(1000000+\theta)^4}\right) &= \frac{1.05^5\theta^5}{4}\left(\frac{1}{(5000+1.05\theta)^4}-\frac{1}{(1000000+1.05\theta)^4}\right)\\ \frac{1}{(5000+\theta)^4}-\frac{1}{(1000000+\theta)^4} &= 1.22015445746\left(\frac{1}{(5000+1.05\theta)^4}-\frac{1}{(1000000+1.05\theta)^4}\right)\\ \frac{1}{(5000+\theta)^4}-\frac{1.22015445746}{(5000+1.05\theta)^4} &= \frac{1}{(1000000+\theta)^4}-\frac{1.22015445746}{(1000000+1.05\theta)^4}\end{aligned}$$

Numerically, we can solve this equation to get $\theta = 826942$.

15. *An insurance company charges a risk charge equal to the square of the average loss amount, divided by 100,000. It purchases excess-of-loss reinsurance of $500,000 over $500,000. The loading on this reinsurance is 25%. The difference between the insurance company's premiums for policies with limit $500,000 and policies with limit $1,000,000 is exactly the reinsurance premium. If the insurer sells 500 policies with limit $500,000 and 500 policies with limit $1,000,000, what is the expected aggregate payment on this portfolio? [Assume that there is a non-zero possibility of losses greater than $500,000 for a given policy.]*

Let x be the average loss amount for a policy with limit $500,000, and let y be the average loss amount for a policy with limit $1,000,000. The insurance company's premiums for these policies are $x+\frac{x^2}{100000}$ and $y+\frac{y^2}{100000}$ respectively. The expected payment on the reinsurance policy is $y-x$, so the reinsurance premium is $1.25(y-x)$. We are given that

$$\begin{aligned}y+\frac{y^2}{100000}-\left(x+\frac{x^2}{100000}\right) &= 1.25(y-x)\\ \frac{y^2-x^2}{100000} &= 0.25(y-x)\\ (y+x)(y-x) &= 25000(y-x)\end{aligned}$$

Since there is a non-zero possibility of losses greater than $500,000 for a given policy, we have $y \neq x$, so we can divide by $y-x$ to get $y+x = 25000$. The expected aggregate payment on the portfolio is $500(y+x) = 500\times 25000 = \$12,500,000$.

16. *An insurer calculates the ILF on the pure premium from $1,000,000 to $2,000,000 on a particular policy is 1.092. A reinsurer offers excess-of-loss reinsurance of $1,000,000 over $1,000,000 for a loading of 30%. The original insurer uses a loading of 20% on policies with limit $1,000,000. If the insurer buys the excess-of-loss reinsurance, what is the loading on its premium for policies with a limit of $2,000,000? [10 mins]*

    Let m be the expected loss on the policy with limit $1,000,000. With a 20% loading, the insurer charges $1.2m$ for the insurance. The expected payment on the reinsurance is $1.092m - m = 0.092m$. With a loading of 30%, the cost of the reinsurance is $0.092m \times 1.3 = 0.1196m$, so the total cost with a limit of $2,000,000 is $1.2m + 0.1196m = 1.3196m$, and the expected payment is $1.092m$, so the loading is $\frac{1.3196m}{1.092m} - 1 = 20.842490842\%$.