ACSC/STAT 4720, Life Contingencies II
FALL 2015
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Sample Final Examination
Model Solutions

1. *An individual aged 42 has a current salary of \$76,000 for the coming year. The salary scale is $s_y = 1.05^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 65.*

   If the individual retires at age 65, the final average salary is $76000\frac{((1.05)^{20}+(1.05)^{21}+(1.05)^{22})}{3} = \$211,901.20$.

2. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 60% of final average salary for an employee who enters the plan at exact age 30, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 60% of the life annuity.*
   - *At age 65, the employee is married to someone aged 63.*
   - *The salary scale is $s_y = 1.04^y$.*
   - *Mortalities are independent and given by $\mu_x = 0.0000016(1.092)^x$. The value of the life annuity is based on $\delta = 0.045$. This gives $\overline{a}_{65} = 19.63036$, $\overline{a}_{63} = 19.83656$ and $\overline{a}_{65,63} = 18.7867$.*
   - *A fixed percentage of salary is payable annually in arrear.*
   - *Contributions earn an annual rate of 7%.*

   Calculate the percentage of salary payable annually to achieve the target replacement rate under these assumptions.

   Suppose the employee has a current salary at age 30 for the coming year of 1. At retirement, the employee's final average salary is $\frac{1.04^{32}+1.04^{33}+1.04^{34}}{3} = 3.650252$. With a replacement ratio of 60%, this leads to an annuity of $0.6 \times 3.650252 = 2.1901512$ and a reversionary annuity of $0.6 \times 2.1901512 = 1.31409072$. For the reversionary annuity, $\overline{a}_{65|63} = \overline{a}_{63} - \overline{a}_{65,63} = 19.83656 - 18.7867 = 1.04986$, so the total EPV of the annuities at age 65 is $2.1901512 \times 19.63036 + 1.31409072 \times 1.04986 = 44.37307$.

   The contributions should therefore be chosen so that the accumulated value of contributions by age 65 is 44.37307. If the employee pays all salary into the pension plan, the accumulated value will be $\frac{1.07^{35}-1.04^{35}}{1.07-1.04} = 224.3497$. The percentage of salary that needs to be paid into the plan is therefore $\frac{44.37307}{224.3497} = 19.78\%$.

3. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
30	1.000000	39	1.350398	48	1.845766	57	2.553877
31	1.033333	40	1.397268	49	1.912422	58	2.649694
32	1.067933	41	1.445983	50	1.981785	59	2.749515
33	1.103853	42	1.496620	51	2.053975	60	2.853522
34	1.141149	43	1.549263	52	2.129115	61	2.961903
35	1.179879	44	1.604000	53	2.207337	62	3.074855
36	1.220103	45	1.660921	54	2.288777	63	3.192585
37	1.261887	46	1.720122	55	2.373580	64	3.315310
38	1.305295	47	1.781702	56	2.461894	65	3.443256

An employee aged 42 and 4 months has 12 years of service, and a current salary of \$106,000 (for the coming year). She has a defined benefit pension plan with $\alpha = 0.02$ and S_{Fin} is the average of her last 3 years' salary. The employee's mortality is given by $\mu_x = 0.00000195(1.102)^x$. The pension benefit is payable monthly in advance. The interest rate is $i = 0.05$. This results in $\ddot{a}_{65}^{(12)} = 17.15373$ and ${}_{22.66666667}p_{42.33333333} = 0.9901951$. There is no death benefit, and there are no exits other than death or retirement at age 65.

(a) Calculate the EPV of the accrued benefit using the projected unit method under the assumption that the employee retires at age 65. [Calculate the salary scale at non-integer ages by linear interpolation.]

We interpolate $s_{42.33333} = \frac{2}{3} \times 1.496620 + \frac{1}{3} \times 1.549263 = 1.514168$. The employee's final average salary is therefore $106000 \times \frac{3.192585+3.315310+3.443256}{1.514168\times 3} = 232211.57$. The accrued benefit is a pension with annual payment rate $0.02 \times 12 \times 232211.57\times = \$55,730.78$. The EPV of this benefit when the employee reaches age 65 is therefore $55730.78 \times 17.15373 = 955990.75$. The current EPV is $955990.75 \times 0.9901951(1.05)^{-22.666667} = \$313,244.67$.

(b) Calculate the employer's contribution for this employee for the year. [${}_{21.66666667}p_{43.33333333} = 0.9903189$.]

In a year's time, the accrued benefit will be a pension with annual payment rate $0.02 \times 13 \times 232211.57\times = \$60,375.01$. The EPV of this is $60375.01 \times 17.15373 \times 0.9903189(1.05)^{-21.666666667} = 356360.35$. The accumulated value of the previous contributions is $313244.67 \times 1.05 = 328906.90$, so this year's contribution is $356360.35 - 328906.90 = \$27,453.45$.

4. *The service table is given below:*

x	l_x	1	2	3
40	*10000.00*	*118.76*	*0*	*0.51*
41	*9880.73*	*112.29*	*0*	*0.58*
42	*9767.86*	*107.16*	*0*	*0.65*
43	*9660.05*	*101.84*	*0*	*0.73*
44	*9557.49*	*96.80*	*0*	*0.82*
45	*9459.86*	*92.02*	*0*	*0.93*
46	*9366.91*	*87.50*	*0*	*1.04*
47	*9278.37*	*83.19*	*0*	*1.18*
48	*9193.99*	*80.11*	*0*	*1.32*
49	*9112.57*	*75.21*	*0*	*1.49*
50	*9035.87*	*71.48*	*0*	*1.68*
51	*8962.71*	*67.92*	*0*	*1.89*
52	*8892.90*	*64.51*	*0*	*2.12*
53	*8826.26*	*61.23*	*0*	*2.39*
54	*8762.64*	*58.07*	*0*	*2.69*
55	*8701.88*	*55.03*	*0*	*3.03*
56	*8643.83*	*52.06*	*0*	*3.41*
57	*8588.36*	*49.18*	*0*	*3.84*
58	*8535.34*	*46.37*	*0*	*4.32*
59	*8484.64*	*43.62*	*0*	*4.86*
60^-	*8484.64*		*1098.84*	
60	*7385.80*	*21.70*	*819.91*	*5.79*
61	*6538.40*	*18.30*	*611.98*	*6.38*
62	*5901.74*	*10.81*	*384.29*	*5.86*
63	*5500.78*	*9.14*	*639.20*	*6.15*
64	*4846.29*	*7.73*	*351.32*	*6.10*
65^-	*4481.14*		*4481.14*	

The salary scale is $s_y = 1.05^y$*. The accrual rate is 0.02. The benefit for employees who withdraw is a deferred annual pension with COLA 2%, starting from age 65. For an individual aged 65, we have* $\ddot{a}_{65} = 12.85$*. The lifetable for an individual who has withdrawn is*

x	l_x	d_x
57	*10000.00*	*7.54*
58	*9992.46*	*8.22*
59	*9984.24*	*8.95*
60	*9975.29*	*9.76*
61	*9965.52*	*10.65*
62	*9954.87*	*11.63*
63	*9943.25*	*12.69*
64	*9930.55*	*13.86*
65	*9916.69*	*15.15*

Calculate the EPV of deferred pension benefits made to an individual aged exactly 57, with 16 years of service, whose salary for the past year was $121,000.

The current final average salary is $121,0001 + 1.05^{-1} + 1.05^{-2}3 = 115329.55$, so if the individual withdraws at age t, then the annual accrued pension benefits are $0.32 \times 115329.55(1.05)^t(1.02)^{8-t} = 43240.62\left(\frac{1.05}{1.02}\right)^t$.

The total EPV of accrued deferred pension benefits is

$$12.85 \times 43240.62 \times \frac{1}{8588.36} \times 9916.69 \left(49.18 \times \frac{1}{9996.23} \times \left(\frac{1.05}{1.02}\right)^{0.5} + 46.37 \times \frac{1}{9988.35} \times \left(\frac{1.05}{1.02}\right)^{1.5} + 43.62 \times \frac{1}{9979.77} \times \left(\frac{1.05}{1.02}\right)^{2.5} + 21.70 \times \frac{1}{9970.41} \times \left(\frac{1.05}{1.02}\right)^{3.5} + 18.30 \times \frac{1}{9959.20} \times \left(\frac{1.05}{1.02}\right)^{4.5} + 10.81 \times \frac{1}{9949.06} \times \left(\frac{1.05}{1.02}\right)^{5.5} + 9.14 \times \frac{1}{9936.90} \times \left(\frac{1.05}{1.02}\right)^{6.5} + 7.73 \times \frac{1}{9923.62} \times \left(\frac{1.05}{1.02}\right)^{7.5} \right) = \$14,368.26$$

5. *A life aged 52 has mortality given in the table below. The yield rate is in another table below*

x	l_x	d_x
52	*10000.00*	*9.66*
53	*9990.34*	*10.25*
54	*9980.09*	*10.88*
55	*9969.22*	*11.57*
56	*9957.65*	*12.31*
57	*9945.33*	*13.12*

Term (years)	*Yield rate*
1	*0.038*
2	*0.041*
3	*0.043*
4	*0.039*
5	*0.038*

Calculate the net annual premium for a 5-year term insurance with benefit \$450,000 sold to this life.

The EPV of benefits is $450000(0.000966(1.038)^{-1}+0.001025(1.041)^{-2}+0.001088(1.043)^{-3}+0.001157(1.039)^{-4}+0.001231(1.038)^{-5}) = \$2,182.41$.

If the premium is P then the EPV of premiums is $P(1+0.999034(1.038)^{-1}+0.998009(1.041)^{-2}+0.996922(1.043)^{-3}+0.995765(1.039)^{-4}) = 4.616504P$

The net premium is then $\frac{2182.41}{4.616504} = \$472.74$.

6. *An insurance company sells* 300 *one-year life insurance policies to lives aged 48. The death benefit is \$530,000, payable at the end of the year to lives which die during the year. The company uses* $q_{48} = 0.00015$ *and* $i = 0.05$ *to calculate the premium for the policy. This results in a net premium of* $530000 \times 0.00015(1.05)^{-1} = \$75.71$.

   However, q_{48} *is an estimated probability based on past data, and the true value is normally distributed with mean 0.00015 and standard deviation 0.00002. The interest rate cannot be fixed, and the actual interest rate obtained is normally distributed with mean 0.05 and standard deviation 0.003.*

   Calculate the expected aggregate profit (at end of year) of the policies, and the variance of this aggregate profit.

   The accrued value of the premiums is $300 \times 75.71(1+I)$, where I is the interest rate. The benefits are $530000D$, where D is the number of policyholders who die. Conditional on the mortality Q_{48}, D follows a binomial distribution with $n = 300$ and $p = Q_{48}$.

   The expected profit is therefore $22713(1+\mathbb{E}I) - 530000\mathbb{E}(D) = 22713(1.05) - 530000 \times 300 \times 0.00015 = -1.35$. (It is non-zero because of rounding errors.)

   To calculate the variance of the profit, we use the law of total variance:

$$
\begin{aligned}
\mathrm{Var}(D) &= \mathbb{E}(\mathrm{Var}(D|Q_{48})) + \mathrm{Var}(\mathbb{E}(D|Q_{48}) \\
&= \mathbb{E}(300Q_{48}(1-Q_{48})) + \mathrm{Var}(300Q_{48}) \\
&= 300 \times 0.00015 - 300(0.00015^2 + 0.00002^2) + 300^2 \times 0.00002^2 \\
&= 0.04502913
\end{aligned}
$$

Since D and I are independent, the variance of profit is therefore

$$
\begin{aligned}
\mathrm{Var}(22713(1+I) - 530000D) &= 22713^2\,\mathrm{Var}(I) + 530000^2\,\mathrm{Var}(D) \\
&= 22713^2 \times 0.003^2 + 530000^2 \times 0.04502913 \\
&= 12648687260
\end{aligned}
$$

7. *An insurance company has a 5-year term insurance policy with a death benefit of \$600,000, sold to a life age 57. Mortality is given in the lifetable below.*

x	l_x	d_x
57	10000.00	11.19
58	9988.81	11.86
59	9976.95	12.59
60	9964.36	13.38
61	9950.97	14.24
62	9936.73	15.18

Interest rates for each year are $i = 0.06$ *in the first year, and in each subsequent year follow a log-normal distribution with* $\mu = -2.8$ *and* $\sigma = 0.4$.

(a) *The company simulates the following standard normal random variables.*

-0.3357003 0.5909843 -1.7381753 -1.1146107 -0.1042812 -0.5100134 -0.1351612 -0.8412991 0.9334425 0.7837861

Using as many of these simulated values as necessary, simulate one set of interest rates, and calculate the EPV of the benefits and the EPV of annual premium P under these simulated interest rates.

The policy is a 5-year policy. The first year's interest rate is known, so there are 4 subsequent years to simulate. If our normal random variable is Z, then $e^{-2.8+0.4Z}$ is a simulated interest rate. This gives us the following interest rates:

$$
\begin{aligned}
I_2 &= e^{-2.8+0.4\times -0.3357003} = 0.05316898 \\
I_3 &= e^{-2.8+0.4\times 0.5909843} = 0.07702646 \\
I_4 &= e^{-2.8+0.4\times -1.7381753} = 0.03034055 \\
I_5 &= e^{-2.8+0.4\times -1.1146107} = 0.03893568
\end{aligned}
$$

At these interest rates the EPV of the benefits is

$$1.06^{-1}(0.001119 + (1.05316898)^{-1}(0.001186 + (1.07702646)^{-1}(0.001259(1.03034055)^{-1}(0.001338+ (1.03893568)^{-1}0.001424))))600000 = \$3,210.96$$

The EPV of premiums P is

$$P(1+1.06^{-1}(0.998881+1.05316898^{-1}(0.997695+(1.07702646)^{-1}(0.996436+0.995097(1.03034055)^{-1})))) = 4.468044P$$

The insurance company simulates 10,000 sets of interest rates, and under its simulations the EPV of the benefits has mean \$3,144.65, and standard deviation \$85.73. The EPV of $a_{57:\overline{5}|}$ *has mean 4.164824 and standard deviation 0.1077111. The covariance of the simuated EPV of benefits and the simulated value of* $a_{57:\overline{5}|}$ *is 9.230819.*

(b) Find a 95% confidence interval for the EPV of the benefits

If the standard deviation of the simulated benefits is \$85.73, then the standard deviation of the mean benefits from 10,000 simulations is $\frac{85.73}{\sqrt{10000}} = 0.8573$. The 95% confidence interval is therefore $3144.65 \pm 1.96 \times 0.8573 = [3142.97, 3146.33]$.

(c) Find a 95% confidence interval for the net premium — that is the value of P *such that the EPV of future loss is zero.*

We want to find the values of P such that there is a 97.5% probability and a 2.5% probability that $a_{57:\overline{5}|}P - B > 0$, where B is the EPV of the benefit. From our simulation, $a_{57:\overline{5}|}P - B$ is normally distributed with mean $4.164824P - 3144.65$ and variance $0.001077111P^2 - 2 \times 0.09230819P + 0.8573$. The probability that $a_{57:\overline{5}|}P - B > 0$ is therefore $\Phi\left(\frac{4.164824P-3144.65}{\sqrt{0.001077111P^2-2\times0.009230819P+0.8573}}\right)$. We set this equal to 0.975 and 0.025 to get the equations

$$\frac{4.164824P - 3144.65}{\sqrt{0.001077111P^2 - 2 \times 0.009230819P + 0.8573}} = \pm 1.96$$

We can solve these:

$$\begin{aligned}
\frac{4.164824P - 3144.65}{\sqrt{0.001077111P^2 - 2 \times 0.009230819P + 0.8573}} &= \pm 1.96 \\
4.164824P - 3144.65 &= \pm 1.96\sqrt{0.001077111P^2 - 2 \times 0.009230819P + 0.8573} \\
(4.164824P - 3144.65)^2 &= 1.96^2(0.001077111P^2 - 2 \times 0.009230819P + 0.8573) \\
17.34162P^2 - 26193.12P + 9888820 &= 0 \\
P &= \frac{26193.12 \pm \sqrt{26193.12^2 - 4 \times 17.34162 \times 9888820}}{2 \times 17.34162}
\end{aligned}$$

This gives the solutions $P = 744.94$ and $P = 765.48$, so the 95% confidence interval is $[744.94, 765.48]$.

8. *An insurance company sells a 5-year annual life insurance policy to a life aged 53, for whom the lifetable below is appropriate.*

x	l_x	d_x
53	*10000.00*	*49.24*
54	*9950.76*	*54.62*
55	*9896.14*	*60.60*
56	*9835.55*	*67.22*
57	*9768.32*	*74.56*
58	*9693.76*	*82.68*

The annual gross premium is \$685. Initial expenses are \$400. The death benefits are \$90,000. Renewal costs are 2% of each subsequent premium. The interest rate is $i = 0.05$

(a) Calculate the profit vector for the policy.

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
0		400			-400
1	685	0.00	34.250	443.16	276.09
2	685	13.70	33.565	494.01	210.85
3	685	13.70	33.565	551.12	153.74
4	685	13.70	33.565	615.10	89.77
5	685	13.70	33.565	686.96	17.91

The profit vector is the last column of this table.

(b) Calculate the discounted payback period of the policy using a risk discount rate $i = 0.07$.

Using a risk discount rate of $i = 0.07$, we get the following partial NPVs:

t	P(in force)	Discounted Pr_t	NPV(t)
0	1	-400.00	-400.00
1	1.000000	250.99	-149.01
2	0.995076	183.26	33.35
3	0.989614	124.20	156.26
4	0.983554	67.36	222.51
5	0.976832	12.47	234.69

So the discounted payback period is 2 years.

9. *An insurance company sells a 5-year endowment insurance policy to a life aged 35 for whom the lifetable below is appropriate.*

x	l_x	d_x
35	*10000.00*	*8.74*
36	*9991.26*	*9.45*
37	*9981.81*	*10.24*
38	*9971.57*	*11.12*
39	*9960.45*	*12.11*
40	*9948.35*	*13.22*

The benefit is \$300,000. The annual premium is \$60,000, and the interest rate is $i = 0.03$. *Initial expenses are \$2,400 and renewal expenses are \$80 at the start of each year after the first. Use a profit test to calculate the reserves at the start of each year. There are no exits other than death or maturity.*

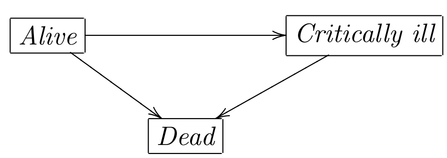
We first perform a profit test without reserves.

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Expected Maturity Benefit	Net Cash Flow
0		2400				-2400
1	60000	0	1800.00	262.20	0.00	61537.80
2	60000	80	1797.60	283.75	0.00	61433.85
3	60000	80	1797.60	307.76	0.00	61409.84
4	60000	80	1797.60	334.55	0.00	61383.05
5	60000	80	1797.60	364.74	299635.26	-238282.40

The reserves at the begining of the fifth year are therefore $238282.40(1.03)^{-1} = 231342.14$. The expected reserve payments at the end of the fourth year are $231342.14 \times \frac{9960.45}{9971.57} = 231084.15$. This makes the net cash flow at end of fourth year with no reserve $61383.05 - 231084.15 = -169701.10$, so the reserve for the fourth year is $169701.10(1.03)^{-1} = 164758.35$. The expected reserve payment at the end of the third year is $164758.35 \times \frac{9971.57}{9981.81} = 164589.33$. This makes a net cash flow of $61409.84 - 164589.33 = -103179.49$ at the end of the third year, so the reserve for the third year is $103179.49(1.03)^{-1} = 100174.26$. The expected reserve payment at the end of the second year is therefore $100174.26 \times \frac{9981.81}{9991.26} = 100079.51$, and the expected cash-flow at the end of the second year is $61433.85 - 100079.51 = -38645.66$. This means that the reserve at the beginning of the second year is $38645.66(1.03)^{-1} = 37520.06$. The expected reserve payment at the end of the first year is therefore $37520.06 \times 0.999126 = 37487.27$. We therefore have the following table:

t	Premium (at $t-1$)	Reserve	Expenses	Interest	Exp. Death Benefits	Exp. Mat. Benefit	Exp. Res. Payment	Net Cash Flow
0		2400						-2400
1	60000	0.00	0	1800.00	262.20	0.00	37487.27	24050.53
2	60000	37520.06	80	1797.60	283.75	0.00	100079.51	0.00
3	60000	100174.26	80	1797.60	307.76	0.00	164589.33	0.00
4	60000	164758.35	80	1797.60	334.55	0.00	231084.15	0.00
5	60000	231342.14	80	1797.60	364.74	299635.26	0.00	0.00

10. *An insurance company offers a 5-year critical illness insurance policy. The policy has 3 states — alive, critically ill, and dead. The possible transitions are as shown in the following diagram:*



Premiums are payable at the start of each year while in the alive state.

For a life aged 37, transitions are as shown in the following lifetable:

age	*Alive*	*Critically Ill*	*Death (direct)*	*Death (critically ill)*	*CI and Death*
37	*10000.00*	*0.00*	*6.95*	*0.00*	*0.03*
38	*9990.20*	*2.82*	*7.47*	*0.03*	*0.04*
39	*9979.47*	*6.01*	*8.03*	*0.08*	*0.03*
40	*9967.71*	*9.63*	*8.66*	*0.15*	*0.04*
41	*9954.78*	*13.75*	*9.36*	*0.22*	*0.03*

At the end of 5 years, the expected number of lives who are critically ill is 18.42.

Initial expenses are 28% of the first premium, and renewal expenses are 4% of subsequent premiums while the life is in the alive state. There are also renewal expenses of \$80 at the start of each year if the life is in the critically ill state. Premiums are payable at the start of each year when the life is in the healthy state. There is a death benfit of \$250,000 at the end of the year in which the life dies, and a benefit of \$100,000 at the end of the year in which the life becomes critically ill. (If the life becomes critically ill and then dies later in the same year, both benefits are payable at the end of the year.) The interest rate is $i = 0.04$. Use a profit test without reserves to determine the premium for this policy which achieves a profit margin of 5% at a risk discount rate of $i = 0.10$.

We first perform a profit test for lives which start the year in the critically ill state, and in the alive state alive state, with the premium set as P.

Critically Ill:

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Net Cash Flow
2	0	80	-3.20	2659.57	-2742.77
3	0	80	-3.20	3327.79	-3410.99
4	0	80	-3.20	3894.08	-3977.28
5	0	80	-3.20	4000.00	-4083.20

Alive:

t	Premium (at $t-1$)	Expenses	Interest	Expected Death Benefits	Expected CI Benefits	Net Cash Flow
0		$0.28P$			$-0.28P$	
1	P	0	$0.04P$	174.50	28.50	$1.04P - 203.00$
2	P	$0.04P$	$0.0384P$	187.93	32.63	$0.9984P - 220.57$
3	P	$0.04P$	$0.0384P$	201.91	37.38	$0.9984P - 239.29$
4	P	$0.04P$	$0.0384P$	218.20	43.24	$0.9984P - 261.44$
5	P	$0.04P$	$0.0384P$	235.82	49.42	$0.9984P - 285.24$

At a risk discount rate of $i = 0.10$, the NPV of the policy is

$-0.28P+(1.04P-203.00)(1.1)^{-1}+0.999020(0.9984P-220.57)(1.1)^{-2}-0.000282\times2742.77(1.1)^{-2}0.997947(0.9984P-239.29)(1.1)^{-3}-0.000601\times3410.99(1.1)^{-3}0.996771(0.9984P-261.44)(1.1)^{-4}-0.000963\times3977.28(1.1)^{-4}0.995478(0.9984P-285.24)(1.1)^{-5} - 0.001375 \times 4083.20(1.1)^{-5} = 3.535186P - 908.6518$.

The NPV of premium payments is $P(1+0.999020(1.1)^{-1}+0.997947(1.1)^{-2}+0.996771(1.1)^{-3}+0.995478(1.1)^{-4}) = 4.161763P$

The profit margin is therefore $\frac{3.535186P-937.607908.6518}{4.161763P}$. Setting this equal to 0.05 gives

$$\begin{aligned}\frac{3.535186P - 908.6518}{4.161763P} &= 0.05\\ 3.535186P - 908.6518 &= 0.20808815P\\ 3.327098P &= 908.6518\\ P &= \$273.11\end{aligned}$$

11. *A policyholder aged 58 buys a 5-year type B universal life insurance policy. The additional death benefit is \$100,000. The policyholder pays a premium of \$7,200 at the start of each year. The lifetable for the policyholder is:*

x	l_x	d_x
58	*10000.00*	*4.12*
59	*9995.88*	*4.41*
60	*9991.46*	*4.73*
61	*9986.73*	*5.08*
62	*9981.65*	*5.46*
63	*9976.19*	*5.87*

The cost of insurance is based on 105% of mortality in the above table and $i = 0.04$*. Expense charges are 1.5% of the account value (after each premium is paid). Assume the credited interest rate is* $i = 0.05$*.*

(a) Calculate the projected account value for the next 5 years.

Cost of insurance for age x is $100000q_x \times \frac{1.05}{1.04}$.

AV_{t-1}	P_t	EC_t	CoI_t	interest	AV_t
0.00	7200	108.00	41.60	352.52	7402.92
7402.92	7200	219.04	44.54	716.97	15056.30
15056.30	7200	333.84	47.80	1093.73	22968.40
22968.40	7200	452.53	51.36	1483.23	31147.74
31147.74	7200	575.22	55.23	1885.86	39603.16

(b) Suppose the insurer earns an interest rate, $i = 0.08$*, and mortality follows the above table, initial expenses are \$1,900 and renewal expenses are 0.5% of account value each year after the first. Suppose there are no surrenders. Calculate the profit margin of this policy at a risk discount rate of* $i = 0.12$*.*

AV_{t-1}	P_t	E_t	I_t	EDB_t	EAV_t	Pr_t
0	0	1900	0	0	0	−1900
0.00	7200	0.00	573.12	44.25	7399.87	331.88
7402.92	7200	73.01	1162.39	50.76	15049.66	591.88
15056.30	7200	111.28	1771.60	58.21	22957.52	900.88
22968.40	7200	150.84	2401.40	66.71	31131.90	1220.36
31147.74	7200	191.74	3052.48	76.36	39581.50	1550.62

This gives the profit vector. The EPV at $i = 0.12$ is

$$\begin{aligned} &331.88 \times 1.000000(1.12)^{-1} + 591.88 \times 0.999588(1.12)^{-2} + 900.88 \times 0.999146(1.12)^{-3} \\ &\quad +1220.36 \times 0.998673(1.12)^{-4} + 1550.62 \times 0.998165(1.12)^{-5} - 1900 = \$1,161.43 \end{aligned}$$

The EPV of premiums received is

$$7200(1 + 0.999588(1.12)^{-1} + 0.999146(1.12)^{-2} + 0.998673(1.12)^{-3} + 0.998165(1.12)^{-4}) = \$29,046.18$$

The profit margin is $\frac{1161.43}{29046.18} = 3.9986\%$.

12. *A life aged 62 buys a 5-year type A universal life insurance policy with death benefit \$600,000. The annual premium is \$7,600. Mortality is as shown in the following table:*

x	l_x	d_x
62	10000.00	12.33
63	9987.67	13.28
64	9974.39	14.33
65	9960.06	15.48
66	9944.58	16.75
67	9927.83	18.15

The credited interest rate is $i = 0.08$. *Cost of insurance is based on mortality in the above table and* $i = 0.05$. *Expense charges are 1.5% of account value (before applying cost of insurance).*

(a) Project the account value for the next 5 years.

Suppose the account value at the start of the year is V. After the premium, let C be the cost of insurance. The account value at the end of the year is $((V + 7600) \times 0.985 - C) \times 1.08$. The additional death benefit is therefore $600000 - ((V + 7600) \times 0.985 - C) \times 1.08$, so the Cost of Insurance is

$$\begin{aligned}
C &= q_x(1.05)^{-1}(600000 - ((V + 7600) \times 0.985 - C) \times 1.08) \\
&= q_x(1.05)^{-1}(600000 - ((V + 7600) \times 1.0638)) + 1.08(1.05)^{-1}q_xC \\
(1 - 1.08(1.05)^{-1}q_x)C &= q_x(1.05)^{-1}(600000 - ((V + 7600) \times 1.0638)) \\
C &= \frac{q_x(1.05)^{-1}(600000 - ((V + 7600) \times 1.0638))}{(1 - 1.08(1.05)^{-1}q_x)}
\end{aligned}$$

Substituting this into the above policy gives

AV_{t-1}	P_t	EC_t	CoI_t	interest	AV_t
0.00	7600	114.00	695.96	543.20	7333.24
7333.24	7600	224.00	740.69	1117.48	15086.04
15086.04	7600	340.29	789.10	1724.53	23281.17
23281.17	7600	463.22	840.84	2366.17	31943.29
31943.29	7600	593.15	896.55	3044.29	41097.88

(b) Assume that the insurance company earns interest $i = 0.075$; *Mortality follows the mortality in the lifetable; Initial expenses are \$2,020; renewal expenses are 0.6% of premiums paid. The surrender charges and surrender rates are:*

Year	Charge	rate
1	\$3,000	4%
2	\$2,200	5%
3	\$1,500	4%
4	\$800	2%
5	0	100%

Calculate the NPV of the policy at a risk discount rate of 10%.

We calculate the profit vector:

AV_{t-1}	P_t	E_t	I_t	EDB_t	EAV_t	ESV_t	Pr_t
0	0	2020	0	0	0	0	−2020
0.00	7600	0.00	570.00	739.80	7031.23	173.12	225.85
7333.24	7600	45.60	1116.57	797.78	14312.68	643.45	250.31
15086.04	7600	45.60	1698.03	862.01	22317.82	870.00	288.65
23281.17	7600	45.60	2312.67	932.52	31255.77	621.90	338.05
31943.29	7600	45.60	2962.33	1010.60	0.00	41028.66	420.76

We then calculate the profit signature:

Year	Probability in force	Pr_t	Π_t
1	1.0000000	225.85	225.85
2	0.9588163	250.31	240.00
3	0.9096644	288.65	262.58
4	0.8720232	338.05	294.79
5	0.8532545	420.76	359.02

At a risk discount rate of 10%, the NPV is

$$225.85(1.1)^{-1} + 240.00(1.1)^{-2} + 262.58(1.1)^{-3} + 294.79(1.1)^{-4} + 359.02(1.1)^{-5} - 2020 = -994.79$$

13. *A life aged 48 has an annual type A Universal life insurance policy that has been in effect for 3 years.*

- *The current account value is \$38,220.*
- *The annual premium is \$6,800.*
- *The expense charge is 1% of account value.*
- *The credited interest rate is* $i = 0.07$.
- *The total death benefit is \$100,000.*
- *The corridor factor requirement is 2.2.*
- *The insurance is priced using mortality rate* $q_{48} = 0.000216$ *and interest* $i = 0.04$.

Calculate the cost of insurance charge for the year.

If the cost of insurance for the year is C, then the account value at year end is

$$((38220 + 6800) * 0.99 - C) \times 1.07$$

If this is less than $\frac{100000}{2.2}$, then the total death benefit is \$100,000, so the additional death benefit is

$$100000 - ((38220 + 6800) * 0.99 - C) \times 1.07$$

which means the cost of insurance is

$$\begin{aligned} C &= 0.000216(100000 - ((38220 + 6800) * 0.99 - C) * 1.07)(1.04)^{-1} \\ C(1 - 0.0002222308) &= 11.69522 \\ C &= \frac{11.69522}{0.9997778} = 11.69782 \end{aligned}$$

This gives an account value of 47677.17, which is more than $\frac{100000}{2.2}$, so the corridor factor applies. Using the corridor factor, the additional death benefit is

$$1.2((38220 + 6800) * 0.99 - C) \times 1.07$$

So the cost of insurance is

$$\begin{aligned} C &= 0.000216 \times 1.2((38220 + 6800) * 0.99 - C) \times 1.07 \times (1.04)^{-1} \\ C(1 + 0.0002666769) &= 11.88574 \\ C &= \frac{11.88574}{1.0002666769} = 11.88257 \end{aligned}$$

[The account value is therefore $((38220 + 6800) * 0.99 - C) \times 1.07 = \$47,676.97$]

14. *An equity-linked insurance policy has the following properties:*

- *Annual premiums are $7,000.*
- *Expense charges are 6% of the first premium and 1.5% of subsequent premiums.*
- *There is a year-end management fee of 0.7% of fund value.*
- *There is a year-end death benefit of 120% of fund value.*
- *Surrenders receive full fund value.*
- *GMMB is the total of the premiums paid.*
- *The annual return is 6%.*
- *The insurer's initial expenses are $500 plus 7.2% of the first premium.*
- *The insurer's renewal expenses are 0.9% of each subsequent premium.*
- *Mortality is given by* $q_x = 0.0002 + 0.00001x$.
- *The policy is sold to a life aged 47.*
- *The policy matures in 5 years.*
- *Surrenders happen at a rate of 4% per year.*

(a) *Project the fund value for the next 5 years.*

t	Alloc. Prem.	Start Value	Int.	Fund before	Mgmt. Charge	Fund
1	6580	0.00	394.80	6974.80	48.82	6925.98
2	6895	6925.98	829.26	14650.23	102.55	14547.68
3	6895	14547.68	1286.56	22729.24	159.10	22570.14
4	6895	22570.14	1767.91	31233.05	218.63	31014.42
5	6895	31014.42	2274.57	40183.98	281.29	39902.69

(b) *Calculate the profit vector for the policy.*

t	Unalloc. Prem.	Expenses	Int.	Mgmt. Charge	Exp. Death Benefit	Pr_t
0	0	1040				-1040
1	420	0	25.20	48.82	0.93	493.09
2	105	63	2.52	102.55	1.98	145.09
3	105	63	2.52	159.10	3.11	200.51
4	105	63	2.52	218.63	4.34	258.81
5	105	63	2.52	281.29	5.67	320.14

(c) If mortality and surrender rates are exactly as in the model, which of the following is the internal rate of return of the policy?

(i) 9.42%

(ii) 10.58%

(iii) 11.04%

(iv) 11.90%

We multiply the profit vector by the probability that the policy is in force to obtain the profit signature:

t	P(in force)	Pr_t	
0	1	-1040.00	-1040.00
1	1	493.09	493.09
2	0.9593568	145.09	139.19
3	0.9203563	200.51	184.54
4	0.8829324	258.81	228.51
5	0.8470217	320.14	271.17

Trying the rates given, we get

(i) NPV$=493.09(1.0942)^{-1}+139.19(1.0942)^{-2}+184.54(1.0942)^{-3}+228.51(1.0942)^{-4}+271.17(1.0942)^{-5}-1040=0.05$

(ii) NPV$=493.09(1.1058)^{-1}+139.19(1.1058)^{-2}+184.54(1.1058)^{-3}+228.51(1.1058)^{-4}+271.17(1.1058)^{-5}-1040=-26.95$

(iii) NPV$=493.09(1.1104)^{-1}+139.19(1.1104)^{-2}+184.54(1.1104)^{-3}+228.51(1.1104)^{-4}+271.17(1.1104)^{-5}-1040=-37.31$

(iv) NPV$=493.09(1.1190)^{-1}+139.19(1.1190)^{-2}+184.54(1.1190)^{-3}+228.51(1.1190)^{-4}+271.17(1.1190)^{-5}-1040=-56.18$

So (i) is the internal rate of return.

15. *For an equity-linked insurance policy with the following properties:*

- *Annual premiums are \$4,000.*
- *Expense charges are 10% of the first premium and 1.2% of subsequent premiums.*
- *There is a year-end management fee of 1% of fund value.*
- *There is a year-end death benefit of 130% of fund value.*
- *Surrenders receive full fund value.*
- *GMMB is the total of the premiums paid.*

- *The insurer's initial expenses are \$300 plus 20% of the first premium.*
- *The insurer's renewal expenses are 0.3% of each subsequent premium.*
- *Mortality is given by* $q_x = 0.0002 + 0.00001x$.
- *The policy is sold to a life aged 54.*
- *The policy matures in 5 years.*
- *Surrenders happen at a rate of 5% per year.*

(a) Use the following random numbers from a standard normal distribution to simulate 5 years of annual returns following a log-normal distribution with $\mu = 0.04$ *and* $\sigma = 0.08$.

```
1.50367034 0.21416629 0.28936230 -0.32615940 1.66506629 0.02024787 -2.64251672 -0.78465126
-0.71987248 0.84399470
```

The random sample is

$$\begin{aligned}
e^{0.08\times 1.50367034+0.04} &= 1.1738555\\
e^{0.08\times 0.21416629+0.04} &= 1.0587969\\
e^{0.08\times 0.28936230+0.04} &= 1.0651855\\
e^{0.08\times -0.32615940+0.04} &= 1.0140044\\
e^{0.08\times 1.66506629+0.04} &= 1.1891102
\end{aligned}$$

(b) Use the simulated returns to project the account values for the next 5 years.

t	Alloc. Prem.	Start Value	Int.	Fund before	Mgmt. Charge	Fund
1	3600	0.00	625.88	4225.88	42.26	4183.62
2	3952	4183.62	478.35	8613.97	86.14	8527.83
3	3952	8527.83	813.50	13293.34	132.93	13160.40
4	3952	13160.40	239.65	17352.05	173.52	17178.53
5	3952	17178.53	3996.00	25126.53	251.27	24875.26

(c) Calculate the NPV for the policy under these returns at a risk discount rate of $i = 10\%$.

t	Unalloc. Prem.	Expenses	Int.	Mgmt. Charge	Exp. Death Benefit	Pr_t
0	0	1100				-1100
1	400	0	69.54	42.26	0.93	510.87
2	48	12	2.12	86.14	1.92	122.34
3	48	12	2.35	132.93	3.00	168.28
4	48	12	0.50	173.52	3.97	206.06
5	48	12	6.81	251.27	5.82	288.25

Since the final account value is more than the total of all premiums, the GMMB has no cost.

The profit signature is then given by

t	Probability in force	Pr_t	Π_t
1	0.9492970	510.87	484.97
2	0.9011558	122.34	110.25
3	0.8554474	168.28	143.95
4	0.8120492	206.06	167.33
5	0.7708450	288.25	222.20

The NPV at the risk discount rate is therefore

$$484.97(1.1)^{-1} + 110.25(1.1)^{-2} + 143.95(1.1)^{-3} + 167.33(1.1)^{-4} + 222.20(1.1)^{-5} - 1100 = -207.59$$

16. *An equity-linked insurance policy has the following properties:*

- *Annual premiums are \$11,000.*
- *Expense charges are 10% of the first premium and 1% of subsequent premiums.*
- *There is a year-end management fee of 1% of fund value.*
- *There is a year-end death benefit of 150% of fund value.*
- *Surrenders receive full fund value.*
- *GMMB is the total of the premiums paid.*
- *The insurer's initial expenses are \$400 plus 10% of the first premium.*
- *The insurer's renewal expenses are 0.5% of each subsequent premium.*
- *Mortality is given by* $q_x = 0.0002 + 0.00003x$.
- *The policy is sold to a life aged 44.*
- *The policy matures in 10 years.*
- *Surrenders happen at a rate of 2% per year.*
- *The insurers' funds receive an annual return of 5%.*
- *Annual returns are log-normally distributed with* $\mu = 0.05$ *and* $\sigma = 0.09$.

The insurance company simulates 5000 sets of annual returns.

The expected fund value at the end of each year (after management charge) is given in the following table.

Year	*Expected fund value*
1	*\$10,777.76*
2	*\$23,738.14*
3	*\$36,545.19*
4	*\$50,891.99*
5	*\$69,002.40*
6	*\$87,803.32*
7	*\$105,490.06*
8	*\$126,203.20*
9	*\$145,938.26*
10	*\$168,093.55*

In 233 of their simulations, the fund value at the end of year 10 was less than \$110,000. The mean fund value at the end of year 10 for these simulations was \$89,492.45. The policy has no reserves.

Calculate the NPV of this policy for the simulated returns at a risk discount rate of 10%.

The management charge is 1% of the fund value before management charge, which is $\frac{1}{99}$ of the fund value after management charge. The death benefit is 50% of the fund value, so the expected death benefit is given by $0.5AVq_x$, where AV is the account value.

The expected GMMB expense is the expected amount by which the fund value is less than \$110,000 at maturity (zero if the fund value is at least \$110,000) times the probability that the life is alive at maturity. From the simulations, the total GMMB expense over the 5,000 simulations is $233 \times (110000 - 89492.45) = 4778259.15$, so the expected GMMB expense for lives which are alive at maturity is $\frac{4778259.15}{5000} = \$955.65$. The probability that a policyholder alive at the start of Year 10 survives to maturity is $1 - q_{53} = 0.9998 - 0.00003 \times 53 = 0.99821$, so the expected GMMB benefit in Year 10 is $0.99821 \times 955.65 = \$953.94$

t	Unalloc. Prem.	Expenses	Int.	Exp. Mgmt. Charge	Exp. Death Payment	Exp. GMMB Benefit	Pr_t
0	0	1500	0.00	0.00			−1500
1	1100	0	55.00	108.87	8.19		158.43
2	110	55	2.75	239.78	18.40		279.13
3	110	55	2.75	369.14	28.87		398.02
4	110	55	2.75	514.06	40.97		530.84
5	110	55	2.75	696.99	56.58		698.16
6	110	55	2.75	886.90	73.32		871.34
7	110	55	2.75	1065.56	89.67		1033.64
8	110	55	2.75	1274.78	109.17		1223.36
9	110	55	2.75	1474.12	128.43		1403.45
10	110	55	2.75	1697.91	150.44	953.94	651.28

We calculate the profit signature:

t	P(in force)	Pr_t	Π_t
1	1.0000000000	158.43	158.43
2	0.9785104000	279.13	273.13
3	0.9574538347	398.02	381.09
4	0.9368222365	530.84	497.31
5	0.9166076736	698.16	639.94
6	0.8968023483	871.34	781.42
7	0.8773985946	1033.64	906.91
8	0.8583888767	1223.36	1050.12
9	0.8397657866	1403.45	1178.57
10	0.8215220429	651.28	535.04

At a risk discount rate of 10%, the NPV of the policy is

$$158.43(1.1)^{-1} + 273.13(1.1)^{-2} + 381.09(1.1)^{-3} + 497.31(1.1)^{-4} + 639.94(1.1)^{-5} + 781.42(1.1)^{-6} + \\ 906.91(1.1)^{-7} + 1050.12(1.1)^{-8} + 1178.57(1.1)^{-9} + 535.04(1.1)^{-10} - 1500 = 1995.57$$

17. *An equity-linked insurance policy has the following properties*

- *Annual premiums are $8,000.*
- *There is a year-end death benefit of 110% of fund value.*
- *Acquisition expenses are $17,000.*
- *The insurer's renewal expenses are $50 at the start of each year after the first.*
- *The policy matures in 15 years.*
- *Reserves on the policy receive an annual return of $i = 0.02$ and are calculated on a reserve basis of 110% of assumed mortality, annual returns of $i = 0.02$ on the fund.*

They plan to use simulation to determine the GMMB, expenses and management charge for the policy. They plan to arrange these so that the NPV of the policy at a risk discount rate of 10% is at least 50% of the acquisition costs, and the probability of making a loss at the risk discount rate is at most 2%. The insurance company simulates 10000 sets of investment returns.

(a) For the first simulation, the company finds that even with the management charge increasing to 100%, the probability of making a loss is still more than 2%. Which of the following changes might solve this problem:

(i) Increasing the expense charges

(ii) Decreasing the expense charges

(iii) Increasing the GMMB

(iv) Decreasing the GMMB

Changing the expense charges is unlikely to solve the problem, because the policy is making a loss even with management charges at 100%. This suggests that the accumulated value of the allocated and unallocated premium less insurer's expenses is less than the Guaranteed minimum maturity benefit. Changing the expense charges will not rectify the situation.

Changing the GMMB should resolve this problem. Increasing the GMMB would only make the problem worse, so the change that might solve the problem is (iv) decreasing the GMMB.

(b) The insurance company finds a combination of charges and benefits which achieves its criteria. However, a new manager suggests that the risk discount rate should be increased to 15%. They are surprised to find that this reduces the management charge required to satisfy the criteria. Why does this happen, and what should be done (if anything) to correct this situation?

This outcome means that for at least some simulations, the NPV at the higher risk discount rate is larger than at the lower risk discount rate. This can only occur because there are large losses in the later years of the policy under these simulations. (These losses are probably associated with the GMMB.) The higher risk discount rate then reduces the impact of these losses, allowing the policy to make a profit.

The problem is that the policy is allowed to make a large loss in later years. They should correct this by increasing the reserves.

(c) The insurance company decides to switch to using a simulation to calculate a 95% quantile reserve in place of the reserves calculated above. If each quantile reserve is to be calculated based on 1,000 simulations, and the overall policy charges and benefits are to be calculated based on 5,000 simulations, how many annual returns do they need to simulate to achieve this? Explain your answer.

For each of the 5,000 simulations to determine policy charges and benefits, they need to simulate 15 years of returns for a total of 75,000 simulations. In addition, for each year, they need to determine reserves. To determine the reserves needed at the start of year t, they need to simulate $16-t$ years of returns. They do this

1000 times for each t, so the total number of simulations needed is $1000(14+13+\cdots+1) = 105000$. This is for a single policy charge and benefit simulation, so it needs to be repeated 5000 times, for a total of $5000 \times 105000 = 525,000,000$ simulations. This is in addition to the 75,000 simulations already performed. Finally, they need to calculate the initial reserve using $1000 \times 15 = 15,000$ simulations. (This does not need to be performed for each of 5,000 simulations because the fund value at the start of Year 1 is zero for all simuations.) The total number of annual returns that need to be simulated is therefore $525000000 + 75000 + 15000 = 525,090,000$ simulations.

18. *An equity-linked insurance policy has the following properties*

- *Annual premiums are \$5,000.*
- *Expense charges are 0.8% of premiums after the first.*
- *There is a year-end management fee of 1.2% of fund value.*
- *There is a year-end death benefit of 130% of fund value.*
- *GMMB is 110% of the total of the premiums paid.*
- *The insurer's renewal expenses are 0.4% of each subsequent premium.*
- *The policy matures in 10 years.*
- *Annual returns are log-normally distributed with $\mu = 0.05$ and $\sigma = 0.3$.*
- *Mortality is given by $q_x = 0.0002 + 0.00003x$.*
- *Reserves on the policy receive an annual return of $i = 0.02$.*

The policy was sold 9 years ago to a life aged 51. The fund value at the beginning of year 10 (before premiums are received) is \$41,205. The company simulates 1000 rates of return. The simulated returns have the following percentiles.

Quantile	i
2.5%	-0.1377533
5%	-0.1010528
95%	*0.2395570*
97.5%	*0.2840839*

Calculate a 95% quantile reserve for the policy at the start of year 9, based on these simulated values.

The end-of-year cashflow is an increasing function of i, so the 95% quantile reserve corresponds to the 5% quantile rate of return, or $i = -0.1010528$. At this rate of return, after the allocated premium of $5000 - 40 = 4960$ is payed, the fund value is \$46,165. After applying the rate of return, the fund value (before management fee) is $46165 \times 0.8989472 = 41499.90$. The management fee is $41499.90 \times 0.012 = 498.00$. The expected death benefit is $(0.0002 + 0.00003 \times 60) \times 0.3 \times (41499.90 - 498.00) = 24.60$. The expected GMMB is $(1 - (0.0002 + 0.00003 \times 60)) \times (55000 - (41499.90 - 498.00)) = 13970.11$. The expected net cashflow is therefore $(40 - 20) \times 0.8989472 + 498.00 - 24.60 - 13970.11 = -13478.73$. The reserve earns interest at $i = 0.02$, so the 95% quantile reserve is $13478.73(1.02)^{-1} = \$13,214.44$.