ACSC/STAT 4720, Life Contingencies II
Fall 2015
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Homework Sheet 2
Model Solutions

Basic Questions

1. *The following is a multiple decrement table giving probabilities of death and surrender for a life insurance policy:*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	*10000.00*	*39.60*	*1.62*
54	*9958.78*	*39.43*	*1.74*
55	*9917.61*	*39.26*	*1.86*
56	*9876.49*	*39.09*	*2.00*
57	*9835.40*	*38.91*	*2.15*
58	*9794.34*	*38.74*	*2.31*
59	*9753.28*	*38.57*	*2.49*
60	*9712.22*	*38.39*	*2.69*
61	*9671.15*	*38.21*	*2.90*
62	*9630.03*	*38.02*	*3.14*

A life insurance policy has a death benefit of \$300,000 payable at the end of the year of death. Premiums are payable at the beginning of each year. Calculate the premium for a 10-year policy sold to a life aged 53 if there is no-payment to policyholders who surrender their policy, and the interest rate is $i = 0.04$.

If the premium is P, then the EPV of premiums received is

$$(1+0.995878(1.04)^{-1}+0.991761(1.04)^{-2}+0.987649(1.04)^{-3}+0.983540(1.04)^{-4}+0.979434(1.04)^{-5}+0.975328(1.04)^{-6}+0.9712$$

The EPV of the benefits is

$$300000(0.000162(1.04)^{-1}+0.000174(1.04)^{-2}+0.000186(1.04)^{-3}+0.000200(1.04)^{-4}+0.000215(1.04)^{-5}+0.000231(1.04)^{-6}+0.$$

The premium is therefore $\frac{544.1657}{8.290407} = \$65.64$.

2. *Update the multiple decrement table from the previous question with the following mortality probabilities*

x	l_x	d_x
53	*10000.00*	*1.25*
54	*9998.75*	*1.33*
55	*9997.41*	*1.42*
56	*9995.99*	*1.51*
57	*9994.48*	*1.61*
58	*9992.87*	*1.72*
59	*9991.15*	*1.84*
60	*9989.31*	*1.97*
61	*9987.34*	*2.10*
62	*9985.24*	*2.25*

using:

(a) UDD in the multiple decrement table.

If the multiple decrement table satisfies UDD, and in a given year, the original decrement probabilities are q^{01} and q^{02}, then the intensity must satisfy $\mu_t^{01} = \frac{q^{01}}{1-t(q^{01}+q^{02})}$ and the individual decrement probability is

$$1 - e^{-\int_0^1 \frac{q^{01}}{1-t(q^{01}+q^{02})}\,dt} = 1 - e^{-\left[-\frac{q^{01}}{q^{01}+q^{02}}\log(1-t(q^{01}+q^{02}))\right]_0^1} = 1 - \left(1-(q^{01}+q^{02})\right)^{\frac{q^{01}}{q^{01}+q^{02}}}$$

Conversely, if the individual decrement probabilities are q' and q'', then the multiple decrement probabilities are the solutions to

$$q' = 1 - \left(1-(q^{01}+q^{02})\right)^{\frac{q^{01}}{q^{01}+q^{02}}}$$
$$q'' = 1 - \left(1-(q^{01}+q^{02})\right)^{\frac{q^{02}}{q^{01}+q^{02}}}$$

This gives

$$\frac{q^{01}}{q^{02}} = \frac{\log(1-q')}{\log(1-q'')}$$
$$1-(q^{01}+q^{02}) = (1-q')(1-q'')$$

If the original multidecrement probabilities are q^{01} and q^{02}, and the probability q^{02} is updated by q'', then the new multidecrement probabilities are the solution to

$$\frac{q^{01}_{\text{new}}}{q^{02}_{\text{new}}} = \frac{\frac{q^{01}}{q^{01}+q^{02}}\log(1-(q^{01}+q^{02}))}{\log(1-q'')}$$
$$1-(q^{01}_{\text{new}}+q^{02}_{\text{new}}) = (1-q'')(1-(q^{01}+q^{02}))^{\frac{q^{01}}{q^{01}+q^{02}}}$$

This gives the updated table

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.60	1.25
54	9959.15	39.43	1.32
55	9918.40	39.26	1.41
56	9877.73	39.10	1.49
57	9837.14	38.92	1.58
58	9796.64	38.75	1.68
59	9756.21	38.58	1.79
60	9715.83	38.41	1.91
61	9675.52	38.23	2.03
62	9635.26	38.04	2.17

(b) UDD in the independent decrements.

If the independent decrements are q' and q'', then the intensities are given by $\mu'_t = \frac{q'}{1-tq'}$ and $\mu''_t = \frac{q''}{1-tq''}$. The multiple decrement probabilities are then the solutions to

$$\begin{aligned}
q^{01} + q^{02} &= 1 - e^{-\int_0^1 \frac{q'}{1-tq'} + \frac{q''}{1-tq''}\,dt} \\
&= 1 - e^{-\left[-\log(1-tq')\right]_0^1 - \left[-\log(1-tq'')\right]_0^1} \\
&= 1 - (1-q')(1-q'') \\
\frac{q^{01}}{q^{02}} &= \frac{q'}{q''}
\end{aligned}$$

Which gives:

$$\begin{aligned}
\frac{q^{02}}{q''}(q' + q'') &= 1 - (1-q')(1-q'') \\
q^{02} &= q'' \frac{q' + q'' - q'q''}{q' + q''} \\
q^{01} &= q' \frac{q' + q'' - q'q''}{q' + q''}
\end{aligned}$$

Given q^{01} and q^{02}, we can therefore solve for q' and q'' to get

$$(1-q'')\left(1-\frac{q^{01}}{q^{02}}q''\right) = 1-(q^{01}+q^{02})$$

$$q'' - \frac{q^{01}+q^{02}}{q^{01}}q'' + \frac{q^{02}(q^{01}+q^{02})}{q^{01}} = 0$$

$$q'' = \frac{\frac{q^{01}+q^{02}}{q^{01}} \pm \sqrt{\left(\frac{q^{01}+q^{02}}{q^{01}}\right)^2 - 4\left(\frac{q^{02}(q^{01}+q^{02})}{q^{01}}\right)}}{2}$$

$$= \frac{q^{01}+q^{02} \pm \sqrt{(q^{01}+q^{02})^2 - 4q^{01}q^{02}(q^{01}+q^{02})}}{2q^{01}}$$

$$q' = \frac{q^{01}+q^{02} \pm \sqrt{(q^{01}+q^{02})^2 - 4q^{01}q^{02}(q^{01}+q^{02})}}{2q^{02}}$$

Substituting in this value into the formulae for the new values of q^{01} and q^{02}, we get

$$q^{02}_{\text{new}} = q''\frac{q'+q''-q'q''}{q'+q''}$$

$$q^{01}_{\text{new}} = q'\frac{q'+q''-q'q''}{q'+q''}$$

This gives the multiple decrement table

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.60	1.25
54	9959.15	39.43	1.32
55	9918.39	39.26	1.41
56	9877.72	39.10	1.49
57	9837.13	38.92	1.58
58	9796.63	38.75	1.69
59	9756.19	38.58	1.80
60	9715.81	38.41	1.92
61	9675.49	38.23	2.03
62	9635.22	38.04	2.17

3. *The mortalities for a husband and wife (whose lives are assumed to be independent) aged 49 and 54 respectively, are given in the following tables:*

x	l_x	d_x
49	10000.00	1.25
50	9998.75	1.31
51	9997.44	1.38
52	9996.06	1.45
53	9994.61	1.53
54	9993.08	1.62
55	9991.46	1.71
56	9989.75	1.81
57	9987.94	1.92
58	9986.01	2.04

x	l_x	d_x
54	10000.00	1.24
55	9998.76	1.33
56	9997.43	1.42
57	9996.01	1.52
58	9994.49	1.63
59	9992.87	1.74
60	9991.12	1.87
61	9989.25	2.01
62	9987.25	2.15
63	9985.09	2.31

The interest rate is $i = 0.04$.

(a) They want to purchase a 10-year last survivor insurance policy with a death benefit of $4,000,000. Annual premiums are payable while both are alive. Calculate the net premium for this policy using the equivalence principle.

If the annual premium is P, then the EPV of the premiums is

$$(1\times 10.999875\times 0.999876(1.04)^{-1}0.999744\times 0.999743(1.04)^{-2}0.999606\times 0.999601(1.04)^{-3}0.999461\times 0.999449(1.04)^{-4}0.99930$$

On the other hand, the EPV of benefits is

$$4000000((1-0.999875)\times(1-0.999876)((1.04)^{-1}-(1.04)^{-2})(1-0.999744)\times(1-0.999743)((1.04)^{-2}-(1.04)^{-3})(1-0.999606)\times($$

The premium is therefore $\frac{8.21}{8.425015} = \$0.97$.

(b) They want to purchase a 10-year reversionary annuity, which will provide an annuity to the wife of $40,000 at the start of each year if the husband is dead and the wife is alive. Calculate the net premium for this policy using the equivalence principle. For the wife, $a_{64} = 19.86$.

As in part (a), the EPV of the premiums is $8.425015P$.

The EPV of the benefits is

$$40000((1-0.999875)\times 0.999876(1.04)^{-1}(1-0.999744)\times 0.999743(1.04)^{-2}(1-0.999606)\times 0.999601(1.04)^{-3}(1-0.999461)\times 0.99$$

The net premium is therefore $\frac{1060.787}{8.425015} = \$125.91$.

4. *A husband is 58; the wife is 72. Their lifetables while both are alive, and the lifetable for the husband if the wife is dead, are given below:*

x	l_x	d_x
72	10000.00	7.92
73	9992.08	8.61
74	9983.47	9.36
75	9974.11	10.17
76	9963.93	11.06
77	9952.87	12.03
78	9940.84	13.08
79	9927.77	14.22
80	9913.54	15.47
81	9898.07	16.82

x	l_x	d_x
58	10000.00	3.49
59	9996.51	3.80
60	9992.70	4.14
61	9988.56	4.51
62	9984.05	4.92
63	9979.13	5.36
64	9973.77	5.85
65	9967.92	6.39
66	9961.53	6.97
67	9954.56	7.61

x	l_x	d_x
58	10000.00	3.94
59	9996.06	4.30
60	9991.76	4.68
61	9987.08	5.10
62	9981.97	5.57
63	9976.41	6.08
64	9970.33	6.63
65	9963.70	7.24
66	9956.46	7.91
67	9948.55	8.64

Calculate the annual premium for a 10-year term insurance policy sold to the husband with death benefit \$200,000. The interest rate is $i = 0.05$. Use the UDD assumption for handling changes to the husband's mortality in the event of the wife's death.

We calculate the mortality for the husband. For each year, the mortality is a combination of the mortality if the wife is alive throughout the year, the mortality if the wife is dead throughout the year, and the mortality if the wife dies during the year. The first two cases are easy to calculate. In the final case, the time of the wifes death is uniformly distributed throughout the year. Suppose the husband's mortality for the year is q if the wife is alive, and q' if the wife is dead. Using the UDD assumption, if the wife dies at time t during the year, the probability that the husband survives the year is

$$(1-tq)\frac{1-q'}{1-tq'}$$

With the wife's death uniformly distributed throughout the year, the probability that the husband survives the year is

$$\begin{aligned}
(1-q')\int_0^1 \frac{1-tq}{1-tq'}\,dt &= (1-q')\int_0^1 \frac{q}{q'} + \frac{1-\frac{q}{q'}}{1-tq'}\,dt \\
&= (1-q')\left(\frac{q}{q'} + \left[-\left(\frac{1}{q'}-\frac{q}{q'^2}\right)\log(1-tq')\right]_0^1\right) \\
&= (1-q')\left(\frac{q}{q'} - \left(\frac{1}{q'}-\frac{q}{q'^2}\right)\log(1-q')\right)
\end{aligned}$$

Using this, we compute the following overall lifetable for the husband:

Age	Both Alive	Husband alive	l_x	d_x
58	10000.00	0.00	10000.00	3.49
59	9988.59	7.92	9996.51	3.80
60	9976.19	16.52	9992.71	4.14
61	9962.71	25.86	9988.57	4.51
62	9948.06	36.00	9984.05	4.92
63	9932.12	47.01	9979.13	5.37
64	9914.79	58.98	9973.76	5.86
65	9895.93	71.97	9967.90	6.40
66	9875.42	86.08	9961.50	6.99
67	9853.11	101.40	9954.52	7.62

From this lifetable, we get

$$A^1_{58:\overline{10}|} = 0.000349(1.05)^{-1}+0.000380(1.05)^{-2}+0.000414(1.05)^{-3}+0.000451(1.05)^{-4}+0.000492(1.05)^{-5}+0.000537(1.05)^{-6}+$$

and thus

$$\ddot{a}_{58:\overline{10}|} = \frac{1-A_{58:\overline{10}|}}{\left(\frac{0.05}{1.05}\right)} = 21(1-0.003958829-0.995454(1.05)^{-10}) = 8.083294$$

The premium is therefore $\frac{200000\times 0.003958829}{8.083294} = \$97.95$.

Standard Questions

5. *A couple want to receive the following:*

- *While both are alive, they would like to receive a pension of \$80,000 per year.*
- *If the husband is alive and the wife is not, they would like to receive a pension of \$45,000 per year.*
- *If the wife is alive and the husband is not, they would like to receive a pension of \$65,000 per year.*
- *When the husband dies: if the wife is still alive, they would like a death benefit of \$130,000; otherwise, they would like a death benefit of \$85,000.*

Construct a combination of insurance and annuity policies that achieve this combination of benefits.

There are a number of ways to achieve this combination. The following are possibilities:

First possibility:

- A life annuity with annual payments \$20,000 while the wife is alive.
- A last survivor annuity of \$45,000 per year.
- A joint life annuity with annual payments \$15,000.
- A life insurance policy of \$85,000 on the husband.
- A contingent life insurance policy of \$45,000 on the husband

While both are alive, they receive payments from all three annuities for a total payment of $20000 + 45000 + 15000 = 80000$.

If the wife is alive, but the husband is dead, they receive the last survivor and the wife's annuity for a total of $20000 + 45000 = 65000$.

If the husband is alive, but the wife is dead, they receive only the last survivor annuity of \$45000.

When the husband dies, if the wife is still alive, they receive both life insurance policies, for a total of $85000 + 45000 = 130000$. If the wife is dead, they receive only the standard life insurance policy for \$85000.

Second possibility:

- A life annuity with annual payments \$35,000 while the wife is alive.
- A life annuity with annual payments \$15,000 while the husband is alive.
- A last survivor annuity of \$30,000 per year.
- A life insurance policy of \$85,000 on the husband.
- A contingent life insurance policy of \$45,000 on the husband

6. *A husband aged 49 and wife aged 46 have the following transition intensities:*

$$\begin{aligned}
\mu_{xy}^{01} &= 0.000002y + 0.000000001x \\
\mu_{xy}^{02} &= 0.0000015x + 0.000000003y \\
\mu_{xy}^{03} &= 0.000042 + 0.000011x + 0.000013y \\
\mu_{x}^{13} &= 0.000004x \\
\mu_{x}^{23} &= 0.000003y
\end{aligned}$$

They want to purchase a reversionary annuity, which will pay a continuous annuity at a rate of \$70,000 per year when the husband is alive and the wife is dead. Premiums are payable continuously while both are alive. Force of interest is $\delta = 0.04$.

(a) Calculate the annual rate of continuous premium.

The rate of transition out of State 0 is $0.000042+0.000012501x+0.000015003y = 0.000042+0.000012501(49+t) + 0.000015003(46+t) = 0.001344687 + 0.000027504t$. The probability that the pair are in State 0 at time t is therefore $e^{-\int_0^t 0.001344687+0.000027504s\,ds} = e^{-(0.001344687t+0.000013752t^2)}$. If premiums are payable at a rate P, then the EPV of premiums received is

$$
\begin{aligned}
P\int_0^\infty e^{-(0.001344687t+0.000013752t^2)}e^{-0.04t}\,dt &= \int_0^\infty e^{-(0.041344687t+0.000013752t^2)}\,dt\\
&= P\int_0^\infty e^{-0.000013752(t^2+3006.449t)}\,dt\\
&= P\int_0^\infty e^{-0.000013752((t+1503.225t)^2-1503.225^2)}\,dt\\
&= Pe^{31.07517}\int_0^\infty e^{-0.000013752(t+1503.225t)^2}\,dt\\
&= Pe^{31.07517}\sqrt{\frac{2\pi}{0.000027504}}\left(1-\Phi(1503.225\sqrt{0.000027504})\right)\\
&= 23.81292P
\end{aligned}
$$

If the wife dies at time t, and the husband is still alive, then the EPV of the benefit is

$$
\begin{aligned}
70000\int_t^\infty e^{-\int_t^s 0.000004(49+u)\,du}e^{-0.04s} &= 70000\int_t^\infty e^{-0.000004(49(s-t)+(s^2-t^2))}e^{-0.04s}\\
&= 70000e^{-0.04t}\int_t^\infty e^{-0.040196(s-t)-0.000004(s^2-t^2))}\,ds
\end{aligned}
$$

The total EPV of the benefit is therefore

$$
\begin{aligned}
&70000\int_0^\infty (0.000002(46+t)+0.000000001(49+t))e^{-(0.041344687t+0.000013752t^2)}\int_t^\infty e^{-0.040196(s-t)-0.000004(s^2-t^2))}\,ds\,dt\\
&= 70000\int_0^\infty (0.000002(46+t)+0.000000001(49+t))e^{-(0.041344687t+0.000013752t^2)}e^{0.000001(4t^2+40196t)}\int_t^\infty e^{-0.000001(4s^2+401\cdots}\\
&= 70000\int_0^\infty (0.000002(46+t)+0.000000001(49+t))e^{-(0.001148687t+0.000009752t^2)}\int_t^\infty e^{0.000004(5024.5)^2}e^{-0.000004(s+5024.5)^2}\cdots\\
&= 70000e^{0.000004(5024.5)^2}\int_0^\infty (0.000002(46+t)+0.000000001(49+t))e^{-(0.001148687t+0.000009752t^2)}\int_t^\infty e^{-0.000004(s+5024.5)^2}\cdots\\
&= 70000e^{100.9824}\int_0^\infty (0.000002(46+t)+0.000000001(49+t))e^{-(0.001148687t+0.000009752t^2)}\sqrt{0.000004\pi}\left(1-\Phi\left(\sqrt{0.000008}\cdots\right.\right.\\
&= 5701.72
\end{aligned}
$$

The annual rate of continuous premium is therefore

$$
\frac{5701.72}{23.81292} = \$239.44
$$

(b) Calculate the policy value after 5 years if both are still alive.

If both are still alive after 5 years, the rate of transition out of State 0 is $0.000042 + 0.000012501x + 0.000015003y = 0.000042 + 0.000012501(54+t) + 0.000015003(51+t) = 0.001482207 + 0.000027504t$. the EPV of premiums is therefore

$$
\begin{aligned}
239.44 \int_0^\infty e^{-(0.001482207t+0.000013752t^2)} e^{-0.04t}\,dt = \int_0^\infty e^{-(0.041482207t+0.000013752t^2)}\,dt \\
&= 239.44 \int_0^\infty e^{-0.000013752(t^2+3016.449t)}\,dt \\
&= 239.44 \int_0^\infty e^{-0.000013752((t+1508.225t)^2-1508.225^2)}\,dt \\
&= 239.44 e^{31.28226} \int_0^\infty e^{-0.000013752(t+1508.225t)^2}\,dt \\
&= 239.44 e^{31.28226} \sqrt{\frac{2\pi}{0.000027504}} \left(1 - \Phi(1508.225\sqrt{0.000027504})\right) \\
&= 239.44 \times 23.73853 \qquad = 5683
\end{aligned}
$$

The EPV of benefits is

$$
\begin{aligned}
&70000 \int_0^\infty (0.000002(51+t) + 0.000000001(54+t)) e^{-(0.041482207t+0.000013752t^2)} \int_t^\infty e^{-0.040216(s-t)-0.000004(s^2-t^2))}\,ds\,dt \\
&= 70000 \int_0^\infty (0.000002(51+t) + 0.000000001(54+t)) e^{-(0.041482207t+0.000013752t^2)} e^{0.000001(4t^2+40216t)} \int_t^\infty e^{-0.000001(4s^2+402} \\
&= 70000 \int_0^\infty (0.000002(51+t) + 0.000000001(54+t)) e^{-(0.001266207t+0.000009752t^2)} \int_t^\infty e^{0.000004(5027)^2} e^{-0.000004(s+5027)^2}\,ds\,d \\
&= 70000 e^{10.054^2} \int_0^\infty (0.000002(51+t) + 0.000000001(54+t)) e^{-(0.001266207t+0.000009752t^2)} \int_t^\infty e^{-0.000004(s+5027)^2}\,ds\,dt \\
&= 70000 e^{101.082916} \int_0^\infty (0.000002(51+t) + 0.000000001(54+t)) e^{-(0.001266207t+0.000009752t^2)} \sqrt{\frac{\pi}{0.000004}} \Big(1 - \Phi\Big(\sqrt{0.000008} \\
&= 6084.22
\end{aligned}
$$

The policy value is therefore $6084.22 - 5683.95 = \$300.27$.