ACSC/STAT 4720, Life Contingencies II
FALL 2016
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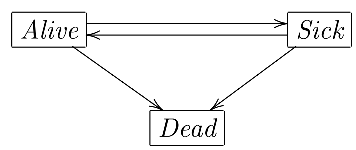
Midterm Examination
Model Solutions

1. *An insurance company is considering a new policy. The policy includes states with the following state diagram:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) Healthy—Sick—Dead

This is possible.

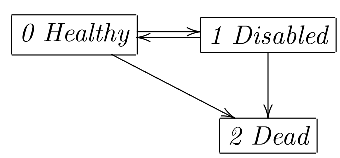
(ii) Healthy—Sick—Healthy—Dead

This is possible.

(iii) Healthy—Sick—Dead—Sick

This is not possible, since it is not possible to transition from dead to sick.

2. *Under a certain model for transition intensities in a disability insurance model, with the following transition diagram:*



you calculate:

${}_{21}p_{59}^{00} = 0.7130$	${}_{21}p_{59}^{01} = 0.0844$	${}_{21}p_{59}^{02} = 0.2026$
$\overline{a}_{59}^{00} = 21.351$	$\overline{a}_{59}^{01} = 1.635$	$\overline{a}_{59}^{02} = 2.014$
$\overline{a}_{80}^{00} = 7.022$	$\overline{a}_{80}^{01} = 1.310$	$\overline{a}_{80}^{02} = 16.668$
$\overline{a}_{80}^{11} = 4.848$	$\overline{a}_{80}^{10} = 2.697$	$\overline{A}_{59}^{02} = 0.08056$
$\overline{A}_{59}^{01} = 0.07442$	$\overline{A}_{80}^{02} = 0.66672$	$\overline{A}_{80}^{12} = 0.69828$

‘ *where 0 is healthy, 1 is disabled, and 2 is dead. Calculate the rate of continuous premium for the following policy for a life aged 59:*

- *Continuous premiums are payable while in the healthy state until age 80.*
- *Continuous benefits at a rate of $80,000 per year are payable up to age 80 while the life is in the disabled state.*
- *a death benefit of $360,000 is payable immediately upon death (regardless of age).*

- *Force of interest is $\delta = 0.04$.*

We calculate

$$\overline{a}^{00}_{59:\overline{21}|} = \overline{a}^{00}_{59} - e^{-0.84}({}_{21}p^{00}_{59}\overline{a}^{00}_{80} + {}_{21}p^{01}_{59}\overline{a}^{10}_{80}) = 21.351 - e^{-0.84}(0.7130\times 7.022 + 0.0844\times 2.697) = 19.09129$$

and

$$\overline{a}^{01}_{59:\overline{21}|} = \overline{a}^{01}_{59} - e^{-0.84}({}_{21}p^{00}_{59}\overline{a}^{01}_{80} + {}_{21}p^{01}_{59}\overline{a}^{11}_{80}) = 1.055126$$

The total EPV of benefits is

$$360000A^{02}_{59} + 80000\overline{a}^{01}_{59:\overline{21}|} = 360000\times 0.08056 + 80000\times 1.055126 = 113411.68$$

The premium is therefore $\frac{113411.68}{19.09129} = \$5940.49$.

3. *The following is a multiple decrement table giving probabilities of surrender (decrement 1) and death (decrement 2) for a life insurance policy:*

x	l_x	$d^{(1)}_x$	$d^{(2)}_x$
36	*10000.00*	*43.30*	*1.10*
37	*9955.60*	*43.11*	*1.22*
38	*9911.27*	*42.93*	*1.34*
39	*9867.00*	*42.74*	*1.48*
40	*9822.78*	*42.56*	*1.63*

A life insurance policy has a death benefit of \$800,000 payable at the end of the year of death. Premiums are payable at the beginning of each year. Calculate the premium for a 4-year policy sold to a life aged 36 if there is no-payment to policyholders who surrender their policy, and the interest rate is $i = 0.04$.

We calculate $\ddot{a}^{00}_{36:\overline{4}|} = 1+0.995560(1.04)^{-1}+0.991127(1.04)^{-2}+0.986700(1.04)^{-3} = 3.750795$ and $A^{02}_{36:\overline{4}} = 0.000110(1.04)^{-1} + 0.000122(1.04)^{-2} + 0.000134(1.04)^{-3} + 0.000148(1.04)^{-4} = 0.0004642016$. The premium is therefore $\frac{0.0004642016\times 800000}{3.750795} = \$99.01$.

4. *A husband is 73; the wife is 82. Their lifetables while both are alive, and the lifetable for the husband if the wife is dead, are given below:*

x	l_x	d_x
73	*10000.00*	*126.67*
74	*9873.33*	*142.76*
75	*9730.57*	*160.62*
76	*9569.96*	*180.34*

x	l_x	d_x
82	*10000.00*	*303.06*
83	*9696.94*	*335.54*
84	*9361.40*	*369.88*
85	*8991.52*	*405.65*

x	l_x	d_x
73	*10000.00*	*576.75*
74	*9423.25*	*631.47*
75	*8791.78*	*684.53*
76	*8107.24*	*733.44*

Calculate the probability that the husband survives for at least 2 years. Use the UDD assumption for handling changes to the husband's mortality in the event of the wife's death.

We use the formula

$$(1-q_d)\left(\frac{q_a}{q_d} + \left(\frac{q_a - q_d}{{q_d}^2}\right)\log(1-q_d)\right)$$

to calculate the probility that the husband survives the year conditional on the wife dying during the year.

This gives the following conditional probabilities:

Year	q_a	q_d	Survival prob if wife dies
1	0.012667	0.057675	0.9643834
2	$\frac{142.76}{9873.33} = 0.01445915$	$\frac{631.47}{9423.25} = 0.06701191$	0.958657

This gives the following probabilities when they are both alive at the start of the year:

Time	P(both alive)	P(H alive, W dead)
1	0.9574109	$0.030306 \times 0.9643834 = 0.0292266$
2	0.9109176	$0.9574109 \times \frac{335.54}{9696.94} \times 0.958657 + 0.0292266 \times \frac{8791.78}{9423.25} = 0.05902739$

So the probability that the husband survives for 2 years is $0.9109176 + 0.05902739 = 0.9699450$

5. *A couple want to receive the following:*

   - *While both are alive, they would like to receive a pension of \$80,000 per year.*
   - *If the husband is alive and the wife is not, they would like to receive a pension of \$65,000 per year.*
   - *If the wife is alive and the husband is not, they would like to receive a pension of \$75,000 per year.*

   Construct a combination of annuity policies that achieve this combination of benefits.

   There are several combinations which work. One is:

   - A last survivor annuity of \$60,000.
   - A life annuity of \$5,000 for the husband.
   - A life annuity of \$15,000 for the wife.

State	Last survivor	Husband	Wife	Total
Both alive	\$60,000	\$5,000	\$15,000	\$80,000
Husband alive	\$60,000	\$5,000	\$0	\$75,000
Wife alive	\$60,000	\$0	\$15,000	\$65,000

6. *A husband aged 49 and wife aged 51 have the following transition intensities:*

$$\begin{aligned}
\mu_{xy}^{01} &= 0.000005y + 0.0000007x \\
\mu_{xy}^{02} &= 0.000005x + 0.0000003y \\
\mu_{xy}^{03} &= 0.000042 + 0.000015x + 0.000012y \\
\mu_{x}^{13} &= 0.000006x \\
\mu_{x}^{23} &= 0.000002y
\end{aligned}$$

   Which of the following expressions gives the probability that after 20 years, both are dead? Justify your answer.

   (i)

$$\int_0^{20} e^{-(0.0019386t+0.000018t^2)}(0.001389 + 0.000027t)\,dt$$

(ii)

$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002893+0.0000057t)e^{-(0.000306(t-s)+0.000003(t^2-s^2))}(0.000306+0.000006t)\,ds\,dt$$
$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002603+0.0000053t)e^{-(0.000098(t-s)+0.000001(t^2-s^2))}(0.000098+0.000002t)\,ds\,dt$$

(iii)

$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002603+0.0000053t)e^{-(0.000306(t-s)+0.000003(t^2-s^2))}(0.000306+0.000006t)\,ds\,dt$$
$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002893+0.0000057t)e^{-(0.000098(t-s)+0.000001(t^2-s^2))}(0.000098+0.000002t)\,ds\,dt$$

(iv)

$$\int_0^{20} e^{-(0.0019386t+0.000018t^2)}(0.001389+0.000027t)\,dt$$
$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002893+0.0000057t)e^{-(0.000306(t-s)+0.000003(t^2-s^2))}(0.000306+0.000006t)\,ds\,dt$$
$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002603+0.0000053t)e^{-(0.000098(t-s)+0.000001(t^2-s^2))}(0.000098+0.000002t)\,ds\,dt$$

(v)

$$\int_0^{20} e^{-(0.0019386t+0.000018t^2)}(0.001389+0.000027t)\,dt$$
$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002603+0.0000053t)e^{-(0.000306(t-s)+0.000003(t^2-s^2))}(0.000306+0.000006t)\,ds\,dt$$
$$+\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002893+0.0000057t)e^{-(0.000098(t-s)+0.000001(t^2-s^2))}(0.000098+0.000002t)\,ds\,dt$$

We calculate:

$$\begin{aligned}
\mu_{xy}^{01} &= 0.000005(51+t)+0.0000007(49+t) = 0.0002893+0.0000057t\\
\mu_{xy}^{02} &= 0.000005(49+t)+0.0000003(51+t) = 0.0002603+0.0000053t\\
\mu_{xy}^{03} &= 0.000042+0.000015(49+t)+0.000012(51+t) = 0.001389+0.000027t\\
\mu_{x}^{13} &= 0.000006(51+t) = 0.000306+0.000006t\\
\mu_{x}^{23} &= 0.000002(49+t) = 0.000098+0.000002t
\end{aligned}$$

We get ${}_tp_{51,49} = e^{-\int_0^t(0.0019386+0.000038t)\,dt} = e^{-(0.0019386t+0.000019t^2)\,dt}$. There are three ways to both be dead, firstly they both die simultaneously, secondly, the husband dies first, and thirdly, the wife dies first. The first way has probability given by

$$\int_0^{20} e^{-(0.0019386t+0.000018t^2)}(0.001389+0.000027t)\,dt$$

For the second way, we integrate over the time s at which the husband dies, and the time t at which the wife dies (with $s<t$).

This is given by

$$\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002893+0.0000057t)e^{-(0.000306(t-s)+0.000003(t^2-s^2))}(0.000306+0.000006t)\,ds\,dt$$

The probability of the third way is

$$\int_0^{20}\int_0^t e^{-(0.0019386s+0.000018s^2)}(0.0002603+0.0000053t)e^{-(0.000098(t-s)+0.000001(t^2-s^2))}(0.000098+0.000002t)\,ds\,dt$$

So the answer is (iv).

7. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 70% of final average salary for an employee who enters the plan at exact age 30, with the following assumptions:*

   - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 50% of the life annuity.*
   - *At age 65, the employee is married to someone aged 62.*
   - *The salary scale is* $s_y = 1.05^y$.
   - *The values of these annuities are* $\overline{a}_{65} = 13.83304$, $\overline{a}_{62} = 14.246231$ *and* $\overline{a}_{\overline{65,62}} = 15.342066$.
   - *A fixed percentage of salary is payable annually in arrear.*
   - *Contributions earn an annual rate of 6%.*

   The company makes contributions which will achieve a replacement rate of 70% under these assumptions.

   However, the employee actually retires at age 60, and at the time is married to someone aged 68. This gives annuity values $\overline{a}_{60} = 14.52447$, $\overline{a}_{68} = 13.306522$ *and* $\overline{a}_{\overline{60,68}} = 15.440938$. *The other assumptions are correct. Calculate the actual replacement ratio.*

   We calculate $\overline{a}_{65|62} = \overline{a}_{\overline{65,62}} - \overline{a}_{65} = 15.342066 - 13.83304 = 1.509026$, so we get $\overline{a}_{65} + 0.5\overline{a}_{65|62} = 13.83304 + 0.5 \times 1.509026 = 14.587553$

   On the other hand,

   $\overline{a}_{60} + 0.5\overline{a}_{60|68} = \frac{14.52447+15.440938}{2} = 14.982704.$

   Now under the new scenario, the employee retires 5 years earlier, so the final average salary is multiplied by 1.05^{-5}. If the first contribution is C and the contributions are a fixed percentage of salary, then the intended accumulated value of the contributions is $C\frac{1.06^{35}-1.05^{35}}{1.06-1.05}$, while with the earlier retirement, the accumulated value is $C\frac{1.06^{30}-1.05^{30}}{1.06-1.05}$. The replacement ratio achieved is therefore $0.7 \times \frac{14.587553(1.06^{30}-1.05^{30})}{14.982704(1.05)^{-5}(1.06^{35}-1.05^{35})} = 56.98\%$.

8. *A company sets up a defined benefit plan with annual benefit* $0.02nS_{Fin}$ *where* n *is the number of years of service and* S_{Fin} *is the average of the employees final 3 years' salary. Payments are annual in advance. An employee aged 52 has 14 years of service, and the employee's current final average salary is $89,000. The salary scale is* $s_y = (1.05)^y$. *The current reserve is* ${}_tV = \$174,000$. *The interest rate is* $i = 0.04$ *and next year's reserve if the employee remains employed is* ${}_{t+1}V = \$142,000$. *The only exits possible in the coming year are withdrawl and death. Assume these exits all happen at the start of the year (immediately*

after contributions are made). The death benefit is 3 times the employee's final average salary. The withdrawl benefit is a deferred pension from age 65, with COLA of 2% per year. For valuing this benefit, we have $\ddot{a}_{65} = 16.6124$ *and* ${}_{13}p_{52} = 0.948712$*. The probability that the individual withdraws this year is* 0.105032*. The probability that the individual dies this year is* 0.00239044*. Calculate the employer's contribution to the pension at the start of the year.*

We first need to calculate the EPV of benefits for individuals who exit this year. The expected death benefits are $89000 \times 3 \times 0.00239044 = 638.2475$. An individual who withdraws this year has final average salary \$89,000. With COLA of 2%, and 14 years of service, the pension is $0.02 \times 14 \times 89000 \times (1.02)^{13} = 32236.68$. We have $\ddot{a}_{65} = 16.6124$, so at age 65, the EPV of the pension will be $32236.68 \times 16.6124 = 535528.58$. The current value is then $535528.58\,{}_{13}p_{52}(1.04)^{-13} = \$305,129.10$. Multiplying this by the probability of withdrawl give $305129.10 \times 0.105032 = \$32048.32$.

We have ${}_{t}V + C_t = (1.04)^{-1}p^{00}_{52\,t+1}V + 32048.32 + 32236.68$ so $C_t = (1.04)^{-1} \times 0.89257756 \times 142000 + 32048.32 + 32236.68 - 174000 = \$12,156.19$

Formulae

Probability of an individual surviving the year their spouse dies at a time unifomly distributed throughout the year.

$$(1 - q_d)\left(\frac{q_a}{q_d} + \left(\frac{q_a - q_d}{{q_d}^2}\right)\log(1 - q_d)\right)$$

where q_a is the probability of dying in the year if the spouse is alive for the whole year and q_d is the probability of dying if the spouse is dead for the whole year.