# ACSC/STAT 4720, Life Contingencies II

Fall 2016

Toby Kenney

Homework Sheet 4

Model Solutions

## Basic Questions

1. *A life aged 52 has mortality given in the table below. The yield rate is in another table below*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *52* | *10000.00* | *51.66* |
| *53* | *9948.34* | *56.61* |
| *54* | *9891.73* | *62.01* |
| *55* | *9829.71* | *67.89* |
| *56* | *9761.82* | *74.28* |
| *57* | *9687.54* | *81.21* |

| *Term (years)* | *Yield rate* |
|---|---|
| *1* | *0.034* |
| *2* | *0.032* |
| *3* | *0.037* |
| *4* | *0.038* |
| *5* | *0.038* |

*Calculate the net annual premium for a 5-year term insurance with benefit \$500,000 sold to this life.*

The EPV of benefits is

$$500000\left(0.005166(1.034)^{-1}+0.005661(1.032)^{-2}+0.006201(1.037)^{-3}+0.006789(1.038)^{-4}+0.007428(1.038)^{-5}\right)=\$13,942.30$$

If the premium is $P$, then the EPV of premiums is

$$P\left(1+0.994834(1.034)^{-1}+0.989173(1.032)^{-2}+0.982971(1.037)^{-3}+0.976182(1.038)^{-4}\right)=4.613260P$$

The premium is therefore

$$P=\frac{13942.30}{4.613260}=\$3,022.22$$

2. *An insurance company sells $N$ one-year life insurance policies to lives aged 45. The death benefit is \$530,000, payable at the end of the year to lives which die during the year. The company uses $q_{45}=0.0004$ and $i=0.06$ to calculate the premium for the policy. This results in a net premium of $530000\times 0.0004(1.06)^{-1}=\$200$.*

   *However, $q_{45}$ is an estimated probability based on past data, and the true value is normally distributed with mean 0.0004 and standard deviation 0.00005. The interest rate cannot be fixed, and the actual interest rate obtained is normally distributed with mean 0.06 and standard deviation 0.01.*

   *Calculate the expected aggregate profit of the policies, and the variance of this aggregate profit. (Calculate the profit at the end of the year.)*

   The total premiums received are $200N$, and the number of individuals who die is binomially distributed with parameters $N$ and $q_{45}$. If the interest rate is $i$, then the value of the premiums at the end of the year is $200N(1+i)$. Conditional on the value of $i$ and $q_{45}$, the expected profit at the end of the year is $200N(1+i)-530000Nq_{45}$, and the variance of the profit is $530000^2Nq_{45}(1-q_{45})$. The overall expected profit is therefore $200N(1.06) - 530000N(0.0004) = 0$. The variance of profit is given by the law of total variance

$$\begin{aligned}
&\mathrm{Var}(200N(1+i) - 530000Nq_{45}) + \mathbb{E}(530000^2 Nq_{45}(1-q_{45})) \\
&= 200^2 N^2 \,\mathrm{Var}(i) + 530000^2 N^2 \,\mathrm{Var}(q_{45}) + 530000^2 N\mathbb{E}(q_{45} - {q_{45}}^2) \\
&= 200^2 \times 0.01^2 N^2 + 530000^2 \times 0.00005^2 N^2 + 530000^2 N(0.0004 - 0.0004^2 - 0.00005^2) \\
&= (4 + 26.5^2)N^2 + N(10600^2 - 212^2 - 26.5^2) \\
&= 706.25N^2 + 112314353.75N
\end{aligned}$$

3. *An insurance company sells a 5-year life-insurance policy with a death benefit of \$450,000 to a life aged 43 for whom the following lifetable is appropriate:*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *43* | *10000.00* | *17.44* |
| *44* | *9982.56* | *19.22* |
| *45* | *9963.35* | *21.17* |
| *46* | *9942.17* | *23.32* |
| *47* | *9918.85* | *25.69* |

*The company calculates the annual premium using the current interest rate of* $i = 0.05$. *This gives a premium of \$910.52. However, the insurance company uses the annual interest rates applicable at the time each year to invest the premiums. These interest rates follow a log-normal distribution with* $\mu = 0.05$ *and* $\sigma = 0.01$. *They simulate the following 5 sets of one-year interest rates:*

| | *Simulation 1* | *Simulation 2* | *Simulation 3* | *Simulation 4* | *Simulation 5* |
|---|---|---|---|---|---|
| *Year 2* | *4.01%* | *4.18%* | *5.22%* | *5.10%* | *2.85* |
| *Year 3* | *6.05%* | *5.26%* | *6.20%* | *6.74%* | *5.72* |
| *Year 4* | *5.47%* | *5.38%* | *2.66%* | *5.03%* | *4.65* |
| *Year 5* | *3.35%* | *6.01%* | *4.32%* | *3.22%* | *4.18* |

*(a) Use these simulated interest rates to estimate the EPV of future loss for this policy.*

In simulation 1, the EPV of the benefits future loss is.

$$\begin{aligned}
&450000\left(0.001744(1.05)^{-1} + 0.001922(1.0401)^{-1}(1.05)^{-1} + \cdots + 0.002569(1.0335)^{-1}(1.0605)^{-1}\cdots(1.05)^{-1}\right) \\
=&\$4,136.73
\end{aligned}$$

The EPV of benefits for the other simulations are \$4129.06 \$4133.45, \$4093.416 and \$4189.81

The EPV of premiums for the first simulation is

$$910.52\left(1 + 0.998256(1.05)^{-1} + 0.996335(1.0401)^{-1}(1.05)^{-1} + 0.994217(1.0605)^{-1}(1.0401)^{-1}(1.05)^{-1} + 0.991885(1.0547)^{-1}\right.$$

The EPV of premiums for the other simulations are \$4,136.00, \$4,118.62, \$4,096.73 and \$4,165.01

This gives the EPV of future loss as $\frac{8.93-6.94+14.83-3.32+24.80}{5} = \$7.66$.

*(b) Construct a 95% confidence interval for the EPV future loss from these simulations.*

The variance of the simulated EPVFL values is

$$\frac{(8.93-7.66)^2+(-6.94-7.66)^2+(14.83-7.66)^2+(-3.32-7.66)^2+(24.80-7.66)^2}{5}=170.145$$

The variance of the mean is therefore $\frac{170.145}{5}=34.029$. A 95% confidence interval for the EPVFL is therefore

$$35.95\pm1.96\sqrt{34.029}=[-\$3.77,19.09]$$

## Standard Questions

4. *An insurance company sells 1000 5-year term policies with a death benefit of $400,000 to lives aged 38 with the following lifetable.*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *38* | *10000.00* | *13.46* |
| *39* | *9986.54* | *15.17* |
| *40* | *9971.37* | *17.09* |
| *41* | *9954.28* | *19.25* |
| *42* | *9935.03* | *21.69* |
| *43* | *9913.34* | *24.43* |

*Using an interest rate of* $i=0.02$*, the insurance company sets the premium to $678.53.*

*(a) Assuming the interest rate is correct, calculate the probability that the present value of loss on these policies exceeds $1,000,000. [You may use a simulation to estimate this probability. Make sure to simulate enough cases that your 95% confidence interval has width at most 0.01.]*

If the number of policyholders who die is 10, the largest possible loss is

$$10\times400000(1.02)^{-1}-990\times678.53\frac{1.02-1.02^{-4}}{0.02}-10\times678.53=\$685,217.06$$

If the number of policyholders who die is 12, the smallest possible loss is $12\times400000(1.02)^{-5}-1000\times678.53\frac{1.02-1.02^{-4}}{0.02}=\$1,085,319.73$, so if 12 or more policyholders die, the loss exceeds 1,000,000. If 10 or fewer die, the loss cannot exceed 1,000,000.

If 11 policyholders die, whether the loss exceeds 1000000 depends on the exact timing of the deaths. The largest possible loss is 1079957.6, For every year the death of a policyholder is delayed, the loss decreases by $400000\frac{0.02}{1.02}+678.53=8521.67$, so in total the deaths need to be delayed by 10 years.

The probability that the deaths are not delayed by more than 10 years is 0.0007992391887.

Probability no. of deaths $\leqslant 10=0.7454389629$

Probability no. of deaths $\geqslant 12=0.1649294503$

Probability no. of deaths $=11=0.08963158679$ Probability no. of deaths $=11$ and loss $>1000000=0.08963158679\times0.0007992391887=7.163707671e-05$

Probability loss exceeds $1000000=7.163707671e-05+0.1649294503=0.1650010874$

*(b) Suppose that the interest rate is either $i = 0$ or $i = 0.04$ with each having probability 0.5. Show that the probability that the loss exceeds $1,000,000 is larger than your answer to part (a). [Hint: it is sufficient to consider the cases where the number of policyholders who die guarantees that the loss exceeds $1,000,000.]*

Even with interest rate $i = 0.04$, if 10 policyholders die, the loss is at least $10 \times 400000(1.04)^{-5} - 1000 \times 678.53\frac{1-1.04^{-5}}{0.04} = 723344.95$

If the interest rate is $i = 0.04$, then if 13 policyholders die, the loss certainly exceeds 1000000. If 10 or fewer policyholders die, the loss cannot exceed 1000000.

On the other hand if $i = 0$, then the loss exceeds 1000000 whenever the number of policyholders who die is at least 11.

For $i = 0$, if 11 or more policyholders die, the loss will exceed 1000000, while if 10 or fewer die, the loss cannot exceed 1000000. Overall, the probability that the loss exceeds 1000000 is at least $\frac{1}{2} \times 0.2545610371 + \frac{1}{2} \times 0.1003529041 = 0.1774569706$

This is larger than the answer to (a).