# ACSC/STAT 4720, Life Contingencies II
Fall 2017

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Homework Sheet 7

Model Solutions

## Basic Questions

1. *An insurance company sells a 5-year annual life insurance policy to a life aged 54, for whom the lifetable below is appropriate.*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *54* | *10000.00* | *12.23* |
| *55* | *9987.77* | *13.47* |
| *56* | *9974.30* | *14.82* |
| *57* | *9959.48* | *16.31* |
| *58* | *9943.16* | *17.95* |

*The annual gross premium is \$1,252. Initial expenses are \$170 plus 30% of the first premium. The death benefits are \$800,000. Renewal costs are 3% of each subsequent premium. The interest rate is* $i = 0.05$

*(a) Calculate the expected net cash-flows associated with this policy (assuming no reserve). [This is the profit vector for the policy.]*

We calculate the following:

| $t$ | Premium (at $t-1$) | Expenses | Interest | Expected Death Benefits | Net Cash Flow |
|---|---|---|---|---|---|
| 0 | | 566.00 | | | −566.00 |
| 1 | 1,320 | 0.00 | 66.00 | 978.4 | 407.60 |
| 2 | 1,320 | 39.60 | 64.02 | 1078.91951857 | 265.50 |
| 3 | 1,320 | 39.60 | 64.02 | 1188.65484295 | 155.77 |
| 4 | 1,320 | 39.60 | 64.02 | 1310.10855988 | 34.31 |
| 5 | 1,320 | 39.60 | 64.02 | 1444.20888329 | −99.79 |

*(b) Which of the following is the internal rate of return of the policy:*

*(i)* $i = 0.12429$

*(ii)* $i = 0.17937$

*(iii)* $i = 0.23581$

*(iv)* $i = 0.24836$

The profit signature is calculated as follows.

| $t$ | $P$(in force) | $\Pr_t$ | $\Pi_t$ |
|---|---|---|---|
| 0 | 1.000000 | −566.00 | −566.00 |
| 1 | 1.000000 | 407.60 | 407.60 |
| 2 | 0.998777 | 265.50048143 | 265.175774341 |
| 3 | 0.997430 | 155.76515705 | 155.364840596 |
| 4 | 0.995948 | 34.31144012 | 34.1724101646 |
| 5 | 0.994316 | −99.78888329 | −99.2216832774 |

We evaluate

$407.60(1+i)^{-1}+265.175774341(1+i)^{-2}+155.364840596(1+i)^{-3}+34.1724101646(1+i)^{-4}-99.2216832774(1+i)^{-5}-566$

for each given value of $i$.

| $i$ | NPV |
|---|---|
| 0.12429 | 81.803322898 |
| 0.17937 | 39.145101864 |
| 0.23581 | 0.003068023 |
| 0.24836 | −8.128575093 |

So the answer is (iii) $i = 0.22125$.

2. *An insurance company sells a 5-year annual life insurance policy to a life aged 32, for whom the lifetable below is appropriate.*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *32* | *10000.00* | *2.14* |
| *33* | *9997.86* | *2.34* |
| *34* | *9995.52* | *2.56* |
| *35* | *9992.96* | *2.81* |
| *36* | *9990.15* | *3.08* |

*The annual gross premium is $190. Initial expenses are $60 plus 20% of the first premium. The death benefits are $680,000. Renewal costs are 2% of each subsequent premium. The interest rate is* $i = 0.06$. *Reserves are calculated on the basis* $i = 0.04$, *with mortality following the table.*

*(a) Calculate the reserves.*

The expected present value of future benefits and future premiums in each year are given below:

| Year | EPV future benefits | EPV premiums | Reserve |
|---|---|---|---|
| 1 | 777.27534572 | 865.497741283 | 0 |
| 2 | 662.98823903 | 702.668021889 | 0 |
| 3 | 530.477868013 | 537.252486758 | 0 |
| 4 | 377.635678002 | 365.188116287 | 12.447561715 |
| 5 | 201.583174812 | 186.2 | 15.383174812 |

*(b) Calculate the profit signature.*

We first calculate the profit vector

| $t$ | Reserves | Premium (at $t-1$) | Expenses | Interest | Expected Death Benefits | Change in Reserves | Net Cash Flow |
|---|---|---|---|---|---|---|---|
| 0 | | | 98.00 | | | | −98.00 |
| 1 | 0 | 190 | 0.00 | 11.400 | 145.52 | 0.00000000000 | 55.88000000 |
| 2 | 0 | 190 | 3.80 | 11.172 | 159.154058969 | 0.00000000000 | 38.217941031 |
| 3 | 0 | 190 | 3.80 | 11.172 | 174.158022794 | 12.447561715 | 10.766415491 |
| 4 | 12.447561715 | 190 | 3.80 | 11.9188537029 | 191.214615089 | 15.383174812 | 3.968625517 |
| 5 | 15.383174812 | 190 | 3.80 | 12.0949904887 | 209.646501804 | 0.0000000 | 4.031663497 |

The profit signature is then calculated as

| $t$ | $P$(in force) | $\mathrm{Pr}_t$ | $\Pi_t$ |
|---|---|---|---|
| 0 | 1.000000 | −98.00 | −98.00 |
| 1 | 1.000000 | 55.88000000 | 55.88 |
| 2 | 0.999786 | 38.217941031 | 38.2097623916 |
| 3 | 0.999552 | 10.766415491 | 10.7615921369 |
| 4 | 0.999296 | 3.968625517 | 3.96583160464 |
| 5 | 0.999015 | 4.031663497 | 4.02769230846 |

*(c) Calculate the profit margin at a risk discount rate of $i = 0.08$.*

At a risk discount rate of $i = 0.08$, the NPV is

$$55.88(1.08)^{-1}+38.2097623916(1.08)^{-2}+10.7615921369(1.08)^{-3}+3.96583160464(1.08)^{-4}+4.02769230846(1.08)^{-5}-98 = 0.69$$

The NPV of premiums received is

$$190\left(1.000000 + 0.999786(1.08)^{-1} + 0.999552(1.08)^{-2} + 0.999296(1.08)^{-3} + 0.999015(1.08)^{-4}\right) = 818.949730942$$

so the profit margin is $\frac{0.6985364851}{818.949730942} = 0.085296625508\%$.

3. *For the policy in Question 2:*

*(a) Calculate the reserves and profit signature for a general premium. [You may assume that $P$ is such that the reserves are zero in Years 1 and 2.]*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| 32 | 10000.00 | 2.14 |
| 33 | 9997.86 | 2.34 |
| 34 | 9995.52 | 2.56 |
| 35 | 9992.96 | 2.81 |
| 36 | 9990.15 | 3.08 |

The reserve in Year 3 is $680000\left(\frac{2.56}{9995.52}(1.04)^{-1} + \frac{2.81}{9995.52}(1.04)^{-2} + \frac{3.08}{9995.52}(1.04)^{-3}\right) - 0.98P\left(1 + \frac{9992.96}{9995.52}(1.04)^{-1} + \frac{9990.15}{9995.52}(1\right.$ $530.477868013 - 2.82764466716P$

The reserve in Year 4 is $680000\left(\frac{2.81}{9992.96}(1.04)^{-1} + \frac{3.08}{9992.96}(1.04)^{-2}\right) - 0.98P\left(1 + \frac{9990.15}{9992.96}(1.04)^{-1}\right) = 377.635678003 - 1.9220427173P$.

The reserve in Year 5 is

$680000\left(\frac{3.08}{9990.15}(1.04)^{-1}\right) - 0.98P = 201.583174811 - 0.98P$

The profit vector is then

| $t$ | Reserves | Prem | Exp. | Interest | Exp. D. Benefits | Exp. Res. Payment | Net Cash Flow |
|---|---|---|---|---|---|---|---|
| 0 | | | $60+0.2P$ | | | | $-60-0.2P$ |
| 1 | 0 | $P$ | 0.00000 | $0.06P$ | 145.52 | 0.00000000000 | $1.06P-145.52$ |
| 2 | 0 | $P$ | $0.02P$ | $0.0588P$ | 159.154 | $530.354-2.8270P$ | $3.86578285668P-689.507768591$ |
| 3 | $530.478-2.828P$ | $P$ | $0.02P$ | $31.829-0.1109P$ | 174.158 | $377.539-1.9215P$ | $10.60955736-0.03695289336P$ |
| 4 | $377.636-1.922P$ | $P$ | $0.02P$ | $22.658-0.0565P$ | 191.215 | $201.526-0.97972P$ | $7.496028783-0.01856528034P$ |
| 5 | $201.583-0.98P$ | $P$ | $0.02P$ | 12.095 | 209.647 | | 4.031663496 |

The profit signature is then calculated as

| $t$ | $P$(in force) | $\mathrm{Pr}_t$ | $\Pi_t$ |
|---|---|---|---|
| 0 | 1.000000 | $-60-0.2P$ | $-60-0.2P$ |
| 1 | 1.000000 | $1.06P-145.52$ | $1.06P-145.52$ |
| 2 | 0.999786 | $3.86578285668P-689.507768591$ | $3.86495557915P-689.360213929$ |
| 3 | 0.999552 | $10.60955736-0.03695289336P$ | $10.6048042783-0.0369363384638P$ |
| 4 | 0.999296 | $7.496028783-0.01856528034P$ | $7.49075157874-0.0185522103826P$ |
| 5 | 0.999015 | 4.031663496 | 4.02769230746 |

*(b) Calculate the premium that gives an internal rate of return of $i=0.12$.*

At risk discount rate $i=0.12$, the NPV is $(1.06P-145.52)(1.12)^{-1}+(3.86495557915P-689.360213929)(1.12)^{-2}+(10.6048042783-0.0369363384638P)(1.12)^{-3}+(7.49075157874-0.0185522103826P)(1.12)^{-4}+(4.02769230746)(1.12)^{-5}-60-0.2P=3.78946667489P-724.888094352$

Setting this equal to zero gives $P=\frac{724.888094352}{3.78946667489}=\$191.29$

4. *For a 5-year term insurance policy sold to a life aged 39, with the following lifetable:*

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| *39* | *10000.00* | 9.64 |
| *40* | *9990.36* | 10.71 |
| *41* | *9979.65* | 11.90 |
| *42* | *9967.75* | 13.23 |
| *43* | *9954.52* | 14.70 |

*an actuary performs the following profit test without reserves:*

| *Year* | *Premium* | *Expenses* | *Interest* | *Expected Death Benefits* | $Pr_t$ |
|---|---|---|---|---|---|
| *0* | | *200* | | | −200 |
| *1* | *900* | *0* | *36.00* | *771.00* | 165.00 |
| *2* | *900* | *18* | *35.28* | *857.90* | 59.38 |
| *3* | *900* | *18* | *35.28* | *954.19* | −36.91 |
| *4* | *900* | *18* | *35.28* | *1061.76* | −144.48 |
| *5* | *900* | *18* | *35.28* | *1181.36* | −264.08 |

*Calculate the reserves needed to ensure that all cash flows are non-negative.*

We work backwards. In the final year, the reserve needed is $264.08(1.04)^{-1}=253.923076923$. The expected reserve payment at the end of Year 4 is therefore $253.923076923\times\frac{9954.52}{9967.75}=253.58604978$. This makes the net cash flow for Year 4 $-398.06604978$. The reserve for Year 4 is therefore $398.06604978(1.04)^{-1}=382.755817096$. The expected reserve payment at the end of Year 3 is therefore $382.755817096\times\frac{9967.75}{9979.65}=382.299408883$.

This makes the net cash flow at the end of Year 3 $-419.209408883$. The reserve for Year 3 is therefore $419.209408883(1.04)^{-1} = 403.08597008$. The expected reserve payment at the end of Year 2 is therefore $403.08597008\frac{9979.65}{9990.36} = 402.653848441$. The net cash flow for Year 2 is therefore $-343.273848441$, so the reserve for Year 2 is $343.273848441(1.04)^{-1} = 330.071008116$. The expected reserve payment at the end of Year 1 is $330.071008116 \times 0.999036 = 329.752819664$. The net cash flow at the end of Year 1 is therefore $-164.752819664$. The reserve for Year 1 is therefore $164.752819664(1.04)^{-1} = 158.416172754$

In summary, the reserves are:

| Year | Reserve |
|---|---|
| 1 | 158.42 |
| 2 | 330.07 |
| 3 | 403.09 |
| 4 | 382.76 |
| 5 | 253.92 |

# Standard Questions

5. *An insurer sells a 5-year disability income protection policy for a life aged 42. The transition probabilities are given in the following table:*

| $x$ | $p_x^{01}$ | $p_x^{02}$ | $p_x^{10}$ | $p_x^{12}$ |
|---|---|---|---|---|
| *42* | *0.002136* | *0.001426* | *0.126260* | *0.082503* |
| *43* | *0.002186* | *0.001497* | *0.123351* | *0.087253* |
| *44* | *0.002241* | *0.001584* | *0.120121* | *0.091034* |
| *45* | *0.002299* | *0.001715* | *0.116980* | *0.096115* |
| *46* | *0.002368* | *0.001860* | *0.113402* | *0.103358* |

*The probability of being in each state at the start of each year is*

| $t$ | ${}_tp_{42}^{00}$ | ${}_tp_{42}^{01}$ |
|---|---|---|
| *1* | 0.996438 | *0.002136* |
| *2* | 0.993031596582 | 0.003864363324 |
| *3* | 0.989697441912 | 0.00527376749426 |
| *4* | 0.986341721701 | 0.00642526842903 |
| *5* | 0.982900107192 | 0.00736818444134 |

*The policy pays a benefit of \$45,000 at the end of any year if the life is disabled at that time (State 1), and pays a death benefit of \$350,000 at the end of the year if the life is dead (State 2). The interest rate is* $i = 0.07$. *Initial expenses are \$400 plus 20% of the first premium. Renewal expenses are 2% of each subsequent premium. The premium is \$1,460 at the start of each year. Use a profit test to calculate the reserves for each year in each state using a reserve rate of* $i = 0.06$ *and calculate the profit margin at a risk discount rate of* $i = 0.08$.

We first perform a profit test in each state without reserves:

**Healthy**

| $t$ | Premium (at $t-1$) | Expenses | Interest | Expected Death Benefits | Expected Disability Benefits | Net Cash Flow |
|---|---|---|---|---|---|---|
| 0 | | 692.00 | | | | −692.00 |
| 1 | 1,460 | 0.00 | 102.200 | 499.10 | 96.120 | 966.98 |
| 2 | 1,460 | 29.20 | 100.156 | 523.95 | 98.370 | 908.636 |
| 3 | 1,460 | 29.20 | 100.156 | 554.40 | 100.845 | 875.711 |
| 4 | 1,460 | 29.20 | 100.156 | 600.25 | 103.455 | 827.251 |
| 5 | 1,460 | 29.20 | 100.156 | 651.00 | 106.560 | 773.396 |

**Disabled**

| $t$ | Premium (at $t-1$) | Expenses | Interest | Expected Death Benefits | Expected Disability Benefits | Net Cash Flow |
|---|---|---|---|---|---|---|
| 2 | 0 | 0 | 0 | 30538.55 | 35522.820 | −66061.370 |
| 3 | 0 | 0 | 0 | 31861.90 | 35498.025 | −67359.925 |
| 4 | 0 | 0 | 0 | 33640.25 | 35410.725 | −69050.975 |
| 5 | 0 | 0 | 0 | 36175.30 | 35245.800 | −71421.100 |

We now work backwards to calculate the reserves:

| Year | Healthy Reserves | Disabled Reserves | Expected Healthy Reserve Payment | Expected Disabled Reserve Payment |
|---|---|---|---|---|
| 5 | 0 | 67378.3962264 | | |
| 4 | 0 | 115161.671588 | 154.902932924 | 53020.3968825 |
| 3 | 0 | 149249.654551 | 258.077306029 | 90844.7088238 |
| 2 | 0 | 173470.236136 | 326.259744848 | 117817.080304 |
| 1 | 0 | N/A | 370.532424386 | N/A |

We add these reserves for the disabled state to the profit test (since reserves in the healthy state are all zero, we do not include them).

**Healthy**

| $t$ | Premium (at $t-1$) | Expenses | Interest | Expected Death Benefits | Expected Disability Benefits | Expected Reserve Payments | Net Cash Flow |
|---|---|---|---|---|---|---|---|
| 0 | | 692.00 | | | −692.00 | | |
| 1 | 1,460 | 0.00 | 102.200 | 499.10 | 96.120 | 370.532 | 596.448 |
| 2 | 1,460 | 29.20 | 100.156 | 523.95 | 98.370 | 326.260 | 582.376 |
| 3 | 1,460 | 29.20 | 100.156 | 554.40 | 100.845 | 258.077 | 617.634 |
| 4 | 1,460 | 29.20 | 100.156 | 600.25 | 103.455 | 154.903 | 672.348 |
| 5 | 1,460 | 29.20 | 100.156 | 651.00 | 106.560 | 0 | 773.396 |

**Disabled**

| $t$ | Reserves | Premium less Exp. | Interest | Expected Death Benefits | Expected Disability Benefits | Expected Disability Reserve | Net Cash Flow |
|---|---|---|---|---|---|---|---|
| 2 | 173470.24 | 0 | 12142.92 | 30538.55 | 35522.820 | 117817.08 | 1734.70 |
| 3 | 149249.65 | 0 | 10447.48 | 31861.90 | 35498.025 | 90844.71 | 1492.50 |
| 4 | 115161.67 | 0 | 8061.32 | 33640.25 | 35410.725 | 53020.40 | 1151.62 |
| 5 | 67378.40 | 0 | 4716.49 | 36175.30 | 35245.800 | 0 | 673.78 |

We now calculate the profit signature:

| $t$ | ${}_{(t-1)}p_{42}^{00}$ | ${}_{(t-1)}p_{42}^{01}$ | $\mathrm{Pr}_t^{(0)}$ | $\mathrm{Pr}_t^{(1)}$ | $\Pi_t$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | −692.00 | | −692.00 |
| 1 | 1 | 0 | 596.447575614 | 00000000000000 | 596.447575614 |
| 2 | 0.996438 | 0.002136 | 582.376303295 | 1734.702362 | 584.007203148 |
| 3 | 0.993031596582 | 0.003864363324 | 617.634345542 | 1492.4965462 | 619.097969171 |
| 4 | 0.989697441912 | 0.00527376749426 | 672.348211088 | 1151.6167165 | 671.494663393 |
| 5 | 0.986341721701 | 0.00642526842903 | 773.396000000 | 673.7839623 | 767.161985018 |

At risk discount rate $i = 0.08$, the NPV is $596.447575614(1.08)^{-1}+584.007203148(1.08)^{-2}+619.097969171(1.08)^{-3}+671.494663393(1.08)^{-4}+767.161985018(1.08)^{-5}-692 = 1868.10442859$

The EPV of premiums is

$1460+1460\times 0.996438(1.08)^{-1}+1460\times 0.993031596582(1.08)^{-2}+1460\times 0.989697441912(1.08)^{-3}+1460\times 0.986341721701(1.08)^{-4} = 6255.56952954$

The profit margin is therefore $\frac{1868.10442859}{6255.56952954} = 29.863059147\%$.