ACSC/STAT 4720, Life Contingencies II

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Homework Sheet 3

Model Solutions

Basic Questions

1. *A life aged 54 wants to buy a 5-year term insurance policy. A life-table based on current-year mortality is:*

x	l_x	d_x
54	*10000.00*	*6.92*
55	*9993.08*	*7.62*
56	*9985.47*	*8.40*
57	*9977.07*	*9.26*
58	*9967.82*	*10.21*

The insurance company uses a single-factor scale function $q(x,t) = q(x,0)(1-\phi_x)^t$ *to model changes in mortality. The insurance company uses the following values for* ϕ_x*:*

x	ϕ_x
54	*0.02*
55	*0.02*
56	*0.025*
57	*0.015*
58	*0.02*

Calculate $A^{1}_{54:\overline{5}|}$ *at interest rate* $i = 0.04$*, taking into account the change in mortality.*

We calculate the following mortalities for this individual:

t	$q(54+t, 2018+t)$
0	0.000692
1	$\frac{7.62}{9993.08}(1-0.02)^1 = 0.000747277115764$
2	$\frac{8.40}{9985.47}(1-0.025)^2 = 0.000799686945131$
3	$\frac{9.26}{9977.07}(1-0.015)^3 = 0.000886985783151$
4	$\frac{10.21}{9967.82}(1-0.02)^4 = 0.000944778187564$

This gives us

$$
\begin{aligned}
A^{1}_{54:\overline{5}|} = {} & 0.000692(1.04)^{-1} \\
& + 0.000747277115764(1-0.000692)(1.04)^{-2} \\
& + 0.000799686945131(1-0.000692)(1-0.000747277115764)(1.04)^{-3} \\
& + 0.000886985783151(1-0.000692)(1-0.000747277115764)(1-0.000799686945131)(1.04)^{-4} \\
& + 0.000944778187564(1-0.000692)(1-0.000747277115764)(1-0.000799686945131)(1-0.00088698578 \\
& = 0.00359631921928
\end{aligned}
$$

2. *Using the lifetable from Question 1, the insurance company now uses the following mortality scale based on both age and year:*

			t			
x	*2018*	*2019*	*2020*	*2021*	*2022*	*2023*
54	0.02	0.015	0.01	0.01	0.02	0.015
55	0.01	0.02	0.025	0.02	0.015	0.02
56	0.02	0.025	0.03	0.025	0.02	0.015
57	0.025	0.02	0.02	0.015	0.01	0.015
58	0.015	0.015	0.025	0.03	0.035	0.025

Use this mortality scale to calculate $A^{1}_{54:\overline{5}|}$ *at interest rate* $i = 0.04$.

We calculate the following mortalities for this individual:

t	$q(54+t, 2018+t)$
0	0.000692
1	$\frac{7.62}{9993.08}(1-0.02) = 0.000747277115764$
2	$\frac{8.40}{9985.47}(1-0.025)(1-0.03) = 0.000795585986438$
3	$\frac{9.26}{9977.07}(1-0.02)(1-0.02)(1-0.015) = 0.000878003706499$
4	$\frac{10.21}{9967.82}(1-0.015)(1-0.025)(1-0.03)(1-0.035) = 0.000920800293485$

[If the current year was 2017, the solution would be:

t	$q(54+t, 2018+t)$
0	0.000692
1	$\frac{7.62}{9993.08}(1-0.01) = 0.000754902392456$
2	$\frac{8.40}{9985.47}(1-0.02)(1-0.025) = 0.000803787903824$
3	$\frac{9.26}{9977.07}(1-0.025)(1-0.02)(1-0.02) = 0.000869089963286$
4	$\frac{10.21}{9967.82}(1-0.015)(1-0.015)(1-0.025)(1-0.03) = 0.0009398842$

]

This gives us

$$
\begin{aligned}
A^{1}_{54:\overline{5}|} = 0.000692(1.04)^{-1} \\
&+ 0.000747277115764(1-0.000692)(1.04)^{-2} \\
&+ 0.000795585986438(1-0.000692)(1-0.000747277115764)(1.04)^{-3} \\
&+ 0.000878003706499(1-0.000692)(1-0.000747277115764)(1-0.000795585986438)(1.04)^{-4} \\
&+ 0.000920800293485(1-0.000692)(1-0.000747277115764)(1-0.000795585986438)(1-0.00087800370 \\
&= 0.00356538439436
\end{aligned}
$$

[For 2017, the solution is

$$
\begin{aligned}
A^{1}_{54:\overline{5}|} = 0.000692(1.04)^{-1} \\
&+ 0.000754902392456(1-0.000692)(1.04)^{-2} \\
&+ 0.000803787903824(1-0.000692)(1-0.000754902392456)(1.04)^{-3} \\
&+ 0.000869089963286(1-0.000692)(1-0.000754902392456)(1-0.000803787903824)(1.04)^{-4} \\
&+ 0.000939884237391(1-0.000692)(1-0.000754902392456)(1-0.000803787903824)(1-0.00086908996 \\
&= 0.00358772233455
\end{aligned}
$$

]

3. *A pensions company has the current mortality scale for 2018:*

x	$\phi(x, 2018)$	$\left.\frac{d\phi(x,t)}{dt}\right\|_{x,t=2018}$	$\left.\frac{d\phi(x+t,t)}{dt}\right\|_{x,t=2018}$
54	*0.015845470*	-0.001202552	0.0029517451
55	*0.006067218*	-0.003357078	0.0002835208
56	*0.019612949*	-0.003639662	-0.0043587062
57	*0.024808173*	-0.007938091	0.0005934601
58	*0.012475802*	-0.003159578	-0.0003959116

Current mortality is given in the lifetable in Question 1. The company assumes that from 2030 onwards, we will have $\phi(x, t) = 0.02$ *for all* x *and* t*. Calculate* $A^{1}_{54:\overline{5}|}$ *at interest rate* $i = 0.04$*, using the average of age-based and cohort-based effects.*

If we are given $\phi(x, 2018) = q$, $\phi'(x, 2018) = r$ and $\phi(x, t) = 0.02$ for all x and t from 2030 onwards, we let $f(t) = \phi(x, 2018 + t)$, then we have

$$\begin{aligned} f(0) &= q \\ f'(0) &= r \\ f(12) &= 0.02 \\ f'(12) &= 0 \end{aligned}$$

For cubic interpolation, we have $f(t) = at^3 + bt^2 + ct + d$, so our equations become

$$\begin{aligned} d &= q \\ c &= r \\ 12^3a + 12^2b + 12c + d &= 0.02 \\ 3 \times 12^2a + 2 \times 12b + c &= 0 \\ 12^2b + 24c + 3d &= 0.06 \\ b &= \frac{0.06 - 24c - 3d}{144} \\ a &= \frac{-0.04 + 12c + 2d}{1728} \end{aligned}$$

Using this, we calculate the coefficients for each age:

x	a	b	c	d
55	-0.0000394389467593	0.000849779291667	-0.003357078	0.006067218
56	-0.00002572340625	0.000614673895833	-0.003639662	0.019612949
57	-0.0000495606168981	0.00122284489583	-0.007938091	0.024808173
58	-0.0000306500763889	0.000683350458333	-0.003159578	0.012475802

This gives us, for example

$$\begin{aligned} \phi(55, 2019) &= -0.0000394389467593 \times 1^3 + 0.000849779291667 \times 1^1 - 0.003357078 \times 1 + 0.006067218 \\ &= 0.00352048034491 \end{aligned}$$

Similarly, we compute the following value of $\phi(x,t)$

x	$\phi(x,2019)$	$\phi(x,2020)$	$\phi(x,2021)$	$\phi(x,2022)$
55	0.00352048034491			
56	0.0165622374896	0.0145865333333		
57	0.0180433662789	0.0134268856481	0.0106613674062	
58	0.00996892438194	0.00864484722222	0.0083196700625	0.00880949244441

For cohort-based values, if we let $g_x(t) = \phi(x+t, 2018+t) = at^3 + bt^2 + ct + d$, with $\phi(x, 2018) = q$, and $\frac{d\phi(x+t,2018+t)}{dt} = r$, then we have

$$\begin{aligned} g(0) &= q \\ g'(0) &= r \\ g(12) &= 0.02 \\ g'(12) &= 0 \end{aligned}$$

Substituting the cubic interpolation, the equations become

$$\begin{aligned} d &= q \\ c &= r \\ 12^3a + 12^2b + 12c + d &= 0.02 \\ 3 \times 12^2a + 2 \times 12b + c &= 0 \\ 12^2b + 24c + 3d &= 0.06 \\ b &= \frac{0.06 - 24c - 3d}{144} \\ a &= \frac{-0.04 + 12c + 2d}{1728} \end{aligned}$$

Using this, we calculate the coefficients for each cohort:

x	a	b	c	d
54	0.0000156897460648	−0.000405404808333	0.0029517451	0.015845470
55	−0.0000141570106481	0.000243012825	0.0002835208	0.006067218
56	−0.00003071676875	0.000734514595833	−0.0043587062	0.019612949
57	0.00000968626574074	−0.0001990802875	0.0005934601	0.024808173
58	−0.0000114579486111	0.000222739391667	−0.0003959116	0.012475802

From this we get

$$\phi(55, 2019) = -0.0000141570106481 \times 1^3 + 0.000243012825 \times 1^2 + 0.0002835208 \times 1 + 0.006067218 = 0.00657959461435$$

and similarly, we get the following values:

x	$\phi(x,2019)$	$\phi(x,2020)$	$\phi(x,2021)$	$\phi(x,2022)$
55	0.00657959461435			
56	0.0159580406271	0.0135878608333		
57	0.0252122390782	0.0252762621759	0.0250583598875	
58	0.0122911718431	0.0124832727778	0.0129833571125	0.0137226771556

Taking the average of the age-based and cohort-based values of $\phi(x,t)$ gives the following:

x	$\phi(x,2019)$	$\phi(x,2020)$	$\phi(x,2021)$	$\phi(x,2022)$
55	0.00505003747965			
56	0.0162601390584	0.0140871970833		
57	0.0216278026786	0.019351573912	0.0178598636469	
58	0.0111300481125	0.01056406	0.0106515135875	0.0112660848

This gives us

$$q(54,2018) = \frac{6.92}{10000.00}$$
$$q(55,2019) = (1-0.00505003747965)\times\frac{7.62}{9993.08} = 0.000758676875838$$
$$q(56,2020) = -0.0162601390584)(1-0.0140871970833)\times\frac{8.40}{9985.47} = 0.000815886130407$$
$$q(57,2020) = -0.0216278026786)(1-0.019351573912)(1-0.0178598636469)\times\frac{9.26}{9977.07} = 0.000874578637737$$
$$q(58,2020) = -0.0111300481125)(1-0.01056406)(1-0.0106515135875)(1-0.0112660848)\times\frac{10.21}{9967.82} = 0.0009803499755$$

This gives us

$$\begin{aligned} A^{1}_{54:\overline{5}|} &= 0.000692(1.04)^{-1} \\ &\quad + 0.000758676875838(1-0.000692)(1.04)^{-2} \\ &\quad + 0.000815886130407(1-0.000692)(1-0.000758676875838)(1.04)^{-3} \\ &\quad + 0.000874578637737(1-0.000692)(1-0.000758676875838)(1-0.000815886130407)(1.04)^{-4} \\ &\quad + 0.000980349975562(1-0.000692)(1-0.000758676875838)(1-0.000815886130407)(1-0.0008745786377 \\ &= 0.00363975504216 \end{aligned}$$

Standard Questions

4. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.8 \qquad \sigma_k = 1.3 \qquad K_{2018} = -4.14$$

And the following values of α_x and β_x:

x	α_x	β_x
53	-4.180251	0.1791691
54	-4.219389	0.1788574
55	-4.320727	0.1780642
56	-4.080177	0.1799758
57	-4.397765	0.1790583
58	-4.008800	0.1836070
59	-4.424434	0.1794805
60	-4.354352	0.1812529

The insurance company simulates the following values of Z_t:

```
0.48683324 -0.69007524 -1.34565369 -0.44229856 -0.01575498 -0.38189150 1.57336437
-0.69746487
```

Using these simulated values, calculate the probability that a life aged exactly 53 at the start of 2018 survives for 8 years.

Using our simulated values of Z_t, we simulate

$$
\begin{aligned}
K_{2019} &= K_{2018} - 0.8 + 1.3 \times 0.48683324 = -4.307116788 \\
K_{2020} &= K_{2019} - 0.8 + 1.3 \times -0.69007524 = -6.0042146 \\
K_{2021} &= K_{2020} - 0.8 + 1.3 \times -1.34565369 = -8.553564397 \\
K_{2022} &= K_{2021} - 0.8 + 1.3 \times -0.44229856 = -9.928552525 \\
K_{2023} &= K_{2022} - 0.8 + 1.3 \times -0.01575498 = -10.749033999 \\
K_{2024} &= K_{2023} - 0.8 + 1.3 \times -0.38189150 = -12.045492949 \\
K_{2025} &= K_{2024} - 0.8 + 1.3 \times 1.57336437 = -10.800119268 \\
K_{2026} &= K_{2025} - 0.8 + 1.3 \times -0.69746487 = -12.506823599
\end{aligned}
$$

This gives us

$$
\begin{aligned}
\log(m(53,2018)) &= -4.180251 + 0.1791691 \times -4.14\\
\log(m(54,2019)) &= -4.219389 + 0.1788574 \times -4.307116788\\
\log(m(55,2020)) &= -4.320727 + 0.1780642 \times -6.0042146\\
\log(m(56,2021)) &= -4.080177 + 0.1799758 \times -8.553564397\\
\log(m(57,2022)) &= -4.397765 + 0.1790583 \times -9.928552525\\
\log(m(58,2023)) &= -4.008800 + 0.1836070 \times -10.749033999\\
\log(m(59,2024)) &= -4.424434 + 0.1794805 \times -12.045492949\\
\log(m(60,2025)) &= -4.354352 + 0.1812529 \times -10.800119268\\
m(53,2018) &= 0.007284466505\\
m(54,2019) &= 0.00680737490119\\
m(55,2020) &= 0.00456259987738\\
m(56,2021) &= 0.00362604921041\\
m(57,2022) &= 0.00207965193838\\
m(58,2023) &= 0.00252276972084\\
m(59,2024) &= 0.00137904355356\\
m(60,2025) &= 0.00181457329574
\end{aligned}
$$

Using the approximation $q(x,t) \approx 1 - e^{-m(x,t)}$, we get

$$
\begin{aligned}
q(53,2018) &\approx 0.007257999085\\
q(54,2019) &\approx 0.006784257211\\
q(55,2020) &\approx 0.004552207031\\
q(56,2021) &\approx 0.003619483033\\
q(57,2022) &\approx 0.002077490961\\
q(58,2023) &\approx 0.002519590212\\
q(59,2024) &\approx 0.00137809311\\
q(60,2025) &\approx 0.001812927953
\end{aligned}
$$

The probability that the life survives 8 years is therefore

$$(1-0.007258)(1-0.006784)(1-0.004552)(1-0.003619)(1-0.002077)(1-0.002520)(1-0.001378)(1-0.001813) = 0.970371$$