ACSC/STAT 4720, Life Contingencies II
Fall 2018
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Homework Sheet 4
Model Solutions

Basic Questions

1. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.1 \qquad \sigma_k = 0.9 \qquad K_{2018} = -3.90 \qquad \alpha_{43} = -2.67 \qquad \beta_{43} = 0.31$$

It estimates that its reserves are adequate in a given year provided $q(43,t) < 0.0016$. Calculate the probability that its reserves are still adequate in 5 years' time. Use UDD to calculate the relation between q_x and m_x.

Recall that $m_x = \frac{q_x}{\int_0^1 {}_tp_x\,dt}$. Under UDD, we have

$$\int_0^1 {}_tp_x\,dt = \int_0^1 (1 - tq_x)\,dt = \left[t - q_x\frac{t^2}{2}\right]_0^1 = 1 - \frac{q_x}{2}$$

Therefore, we have that $q(43,t) < 0.0016$ if and only if $m(43,t) < \frac{0.0016}{0.9992} = 0.00160128102482$.

The Lee-Carter model gives $\log(m(43,2023)) = \alpha_{43} + \beta_{43}K_{2023} = -2.67 + 0.31K_{2023}$. We therefore have $m(43,t) < 0.00160128102482$ if

$$\begin{aligned} -2.67 + 0.31K_{2023} &< \log(0.00160128102482) = -6.43695132957 \\ K_{2023} &< -12.1514559018 \end{aligned}$$

We have $K_{2023} = K_{2018} + 5c + \sigma_k(Z_{2019} + Z_{2020} + Z_{2021} + Z_{2022} + Z_{2023}) = -3.90 - 0.5 + 0.9(Z_{2019} + Z_{2020} + Z_{2021} + Z_{2022} + Z_{2023})$. Therefore we have $K_{2023} < -12.1514559018$ if and only if $Z_{2019} + Z_{2020} + Z_{2021} + Z_{2022} + Z_{2023} < \frac{-12.1514559018-(-4.40)}{0.9} = -8.61272877978$. Since each Z_t is i.i.d. standard normal. $Z_{2019} + Z_{2020} + Z_{2021} + Z_{2022} + Z_{2023}$ is normal with mean 0 and variance 5, so $P(Z_{2019} + Z_{2020} + Z_{2021} + Z_{2022} + Z_{2023} < -8.61272877978) = \Phi\left(-\frac{8.61272877978}{\sqrt{5}}\right) = \Phi(-3.85172940467) = 0.00005864329$.

2. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$$
\begin{array}{cccc}
K^{(1)}_{2018} = -2.48 & K^{(2)}_{2018} = 0.33 & c^{(1)} = -0.32 & c^{(2)} = -0.01 \\
\sigma_{k_1} = 0.8 & \sigma_{k_2} = 0.14 & \rho = 0.3 & \overline{x} = 48
\end{array}
$$

(a) Use this scale to calculate the median value of $q(28, 2024)$*.*

Under the CBD model, we have $\log\left(\frac{q(28,2024)}{1-q(28,2024)}\right) = K^{(1)}_{2024} + K^{(2)}_{2024}(28 - 48)$. Since $\log\left(\frac{q(28,2024)}{1-q(28,2024)}\right)$ is an increasing function of $q(28, 2024)$, the median value of $q(28, 2024)$ corresponds to the median value of $K^{(1)}_{2024} + K^{(2)}_{2024}(28 - 48) = K^{(1)}_{2024} - 20K^{(2)}_{2024}$. $K^{(1)}_{2024}$ is normally distributed with mean $K^{(1)}_{2018} + 6c^{(1)}$, and $K^{(2)}_{2024}$ has mean $K^{(2)}_{2018} + 6c^{(2)}$. Therefore the mean of $K^{(1)}_{2024} - 20K^{(2)}_{2024}$ is $K^{(1)}_{2018} + 6c^{(1)} - 20(K^{(2)}_{2018} + 6c^{(2)}) = -2.48 + 6 \times -0.32 - 20(0.33 + 6 \times -0.01) = -9.8$. Since $K^{(1)}_{2024} - 20K^{(2)}_{2024}$ is normally distributed, the mean is the median, so the median value of $q(28, 2024)$ is $\frac{e^{-9.8}}{1+e^{-9.8}} = 0.0000554485247228$.

(b) For a life aged 31, how many years will it be until the life's mortality exceeds 0.001 with probability at least 0.4? [Remember that the life's age increases by 1 each year.]

If $q(x, t) > 0.001$, we have $\log\left(\frac{q(x,t)}{1-q(x,t)}\right) > \log\left(\frac{0.001}{0.999}\right) = -6.90675477865$, so we want to find when $K^{(1)}_{2018+t} + (31+t-48)K^{(2)}_{2018+t}$ has probability 0.4 of being larger than -6.90675477865. Since $K^{(1)}_{2018+t} + (31+t-48)K^{(2)}_{2018+t}$ is normally distributed, we have that $\Phi^{-1}(0.4) = -0.2533471$, so we need that

$$
\frac{-\mathbb{E}(K^{(1)}_{2018+t} + (31 + t - 48)K^{(2)}_{2018+t}) - 6.90675477865}{\sqrt{\mathrm{Var}(K^{(1)}_{2018+t} + (31 + t - 48)K^{(2)}_{2018+t})}} > -0.2533471
$$

We need to find the mean and variance of $K^{(1)}_{2018+t} + (31 + t - 48)K^{(2)}_{2018+t}$. We have that $K^{(1)}_{2018+t} = K^{(1)}_{2018} + tc^{(1)} + \sigma_{k_1} \sum_{i=1}^{t} Z^{(1)}_{2018+i}$ and $K^{(2)}_{2018+t} = K^{(2)}_{2018} + tc^{(2)} + \sigma_{k_2} \sum_{i=1}^{t} Z^{(2)}_{2018+i}$. Thefore we get

$$
K^{(1)}_{2018+t}+(31+t-48)K^{(2)}_{2018+t} = K^{(1)}_{2018}+(t-17)K^{(2)}_{2018}+tc^{(1)}+t(t-17)c^{(2)}+\sum_{i=1}^{t}(\sigma_{k_1}Z^{(1)}_{2018+i}+(t-17)\sigma_{k_2}Z^{(2)}_{2018+
$$

This has mean

$$
-2.48 + 0.33(t - 17) - 0.32t - 0.01t(t - 17)
$$

The variance of $\sigma_{k_1} Z^{(1)}_{2018+i} + \sigma_{k_2}(t - 17)Z^{(2)}_{2018+i}$ is

$$
\sigma^2_{k_1} + (t-17)^2\sigma^2_{k_2} + 2(t-17)\rho\sigma_{k_1}\sigma_{k_2} = 0.64 + 0.0196(t-17)^2 + 0.0672(t-17)
$$

so the variance of $\sum_{i=1}^{t}(\sigma_{k_1} Z^{(1)}_{2018+i} + (t-17)\sigma_{k_2} Z^{(2)}_{2018+i})$ is

$$(0.64 + 0.0196(t-17)^2 + 0.0672(t-17))t$$

We therefore need to solve

$$\frac{2.48 - 0.33(t-17) + 0.32t + 0.01t(t-17) - 6.90675477865}{\sqrt{(0.64 + 0.0196(t-17)^2 + 0.0672(t-17))t}} > -0.2533471$$

$$\frac{0.01t^2 - 0.18t - 4.42675477865}{\sqrt{(0.64 + 0.0196(t-17)^2 + 0.0672(t-17))t}} > -0.2533471$$

Since the quantities are both negative, we can square both sides to get

$$(0.01t^2 - 0.18t - 4.42675477865)^2 < 0.2533471^2(0.64 + 0.0196(t-17)^2 + 0.0672(t-17))t$$

$$0.0001t^4 - 0.0036t^3 - 0.0766675477865t^2 + 0.796815860157t + 19.5961578703 < 0.00125802116034t^3 - 0.01707$$

$$0.0001t^4 - 0.00485802116034t^3 - 0.0595944034676t^2 + 0.465494164766t + 19.5961578703 < 0$$

Numerically, we find that this first happens when $t = 16$. Therefore it will be 16 years before this happens.

Standard Questions

3. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.3 \qquad \sigma_k = 1.2 \qquad K_{2018} = -4.02$$

And the following values of α_x and β_x:

x	α_x	β_x
42	-4.466353	*0.2410742*
43	-4.399855	*0.1913984*
44	-4.357340	*0.1789869*
45	-4.296188	*0.1671459*
46	-4.259301	*0.1891794*
47	-4.210775	*0.1092111*

Using the approximation $m(x,t) \approx q(x,t)$, calculate the probability that a life aged 43 survives for three years under this model.

The probability of surviving for three years is $\mathbb{E}\left((1-q(43,2018))(1-q(44,2019))(1-q(45,2020))\right)$ Using the approximation $m(x,t) \approx q(x,t)$, we have $\log(q(x,t)) = \alpha_x +$

$\beta_x K_t$. We have that

$$\begin{aligned}
\log(q(43,2018)) &= -4.399855 + 0.1913984 \times -4.02 = -5.169276568\\
\log(q(44,2019)) &= -4.357340 + 0.1789869 \times (-4.02 - 0.3 + 1.2Z_{2019})\\
&= -5.130563408 + 0.21478428Z_{2019}\\
\log(q(45,2020)) &= -4.296188 + 0.1671459 \times (-4.02 - 0.6 + 1.2(Z_{2019} + Z_{2020}))\\
&= -5.068402058 + 0.20057508(Z_{2019} + Z_{2020})
\end{aligned}$$

The probability we want to calculate is

$$\begin{aligned}
&(1 - q(43,2018))(1 - \mathbb{E}(q(44,2019)) - \mathbb{E}(q(45,2020)) + \mathbb{E}(q(44,2019)q(45,2020)))\\
=&0.994311317294 \times (1 - \mathbb{E}(q(44,2019)) - \mathbb{E}(q(45,2020)) + \mathbb{E}(q(44,2019)q(45,2020)))
\end{aligned}$$

We know that $q(44,2019)$ is log-normal with $\mu = -5.130563408$ and $\sigma^2 = 0.21478428^2 = 0.0461322869351$. This gives $\mathbb{E}(q(44,2019)) = e^{-5.130563408+\frac{0.0461322869351}{2}} = 0.00605120856227$. Similarly $q(45,2020)$ is log-normal with $\mu = -5.068402058$ and $\sigma^2 = 2 \times 0.20057508^2 = 0.080460725434$. This gives $\mathbb{E}(q(45,2020)) = e^{-5.068402058+\frac{0.080460725434}{2}} = 0.00655077644408$. Finally, $q(44,2019)q(45,2020)$ is log-normal with $\mu = -5.130563408 - 5.068402058 = -10.198965466$ and $\sigma^2 = (0.21478428 + 0.20057508)^2 + 0.20057508^2 = 0.212753760657$. This gives $\mathbb{E}(q(44,2019)) = e^{-10.198965466+\frac{0.212753760657}{2}} = 0.0000413851438515$. Therefore, the probability is

$$0.994311317294 \times (1 - 0.00605120856227 - 0.00655077644408 + 0.0000413851438515) = 0.981822170699$$

4. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$$\begin{array}{cccc}
K^{(1)}_{2018} = -5.04 & K^{(2)}_{2018} = 0.16 & c^{(1)} = -0.2 & c^{(2)} = 0.01\\
\sigma_{k_1} = 0.6 & \sigma_{k_2} = 0.04 & \rho = 0.2 & \overline{x} = 45
\end{array}$$

A husband aged 36 and a wife aged 48 purchase a last survivor insurance contract. The contract has a special clause allowing the wife to surrender the contract for a fixed price upon the death of the husband. The company calculates that the value of this clause is $50000(0.00324 - q(63,2033))_+$ *if the husband dies in 2033. They therefore want to estimate the quantity* $50000q(51,2033)(0.00324 - q(63,2033))_+$. *Calculate the expected value of this quantity.*

We have that $\log\left(\frac{q(51,2033)}{1-q(51,2033)}\right) = K^{(1)}_{2033} + 6K^{(2)}_{2033}$ and $\log\left(\frac{q(63,2033)}{1-q(63,2033)}\right) = K^{(1)}_{2033} + 18K^{(2)}_{2033}$ Recalling that $K^{(1)}_{2033} = K^{(1)}_{2018} + 15c^{(1)} + \sigma_{k_1}\sum_{i=1}^{15} Z^{(1)}_{2018+i} = \sigma_{k_1}\sum_{i=1}^{15} Z^{(1)}_{2018+i} - 8.04$ and $K^{(2)}_{2033} = K^{(2)}_{2018} + 15c^{(2)} + \sum_{i=1}^{15} Z^{(2)}_{2018+i} =$

$\sigma_{k_2} \sum_{i=1}^{15} Z^{(2)}_{2018+i} + 0.31$ We have that $\sigma_{k_1} Z_t^{(1)} + a\sigma_{k_1} Z_t^{(2)}$ is normal with mean 0 and variance ${\sigma_{k_1}}^2 + a^2{\sigma_{k_1}}^2 + 2a\rho\sigma_{k_1}\sigma_{k_2} = 0.36 + 0.0016a^2 + 0.0096a$, so $\sum_{t=2019}^{2033} \sigma_{k_1} Z_t^{(1)} + a\sigma_{k_1} Z_t^{(2)}$ is normal with mean 0 and variance $15(0.36 + 0.0016a^2 + 0.0096a) = 5.4 + 0.024a^2 + 0.144a$ Finally, the covariance of $\sum_{t=2019}^{2033} \sigma_{k_1} Z_t^{(1)} + a_1\sigma_{k_1} Z_t^{(2)}$ and $\sum_{t=2019}^{2033} \sigma_{k_1} Z_t^{(1)} + a_2\sigma_{k_1} Z_t^{(2)}$ is

$$\begin{aligned}
&\frac{1}{2}\left(\mathrm{Var}\left(\sum_{t=2019}^{2033} 2\sigma_{k_1} Z_t^{(1)} + (a_1+a_2)\sigma_{k_1} Z_t^{(2)}\right) - \mathrm{Var}\left(\sum_{t=2019}^{2033} \sigma_{k_1} Z_t^{(1)} + a_1\sigma_{k_1} Z_t^{(2)}\right)\right. \\
&\qquad\qquad\qquad\qquad\left. - \mathrm{Var}\left(\sum_{t=2019}^{2033} \sigma_{k_1} Z_t^{(1)} + a_2\sigma_{k_1} Z_t^{(2)}\right)\right) \\
=&\frac{1}{2}\left(4\left(5.4 + 0.024\left(\frac{a_1+a_2}{2}\right)^2 + 0.144\left(\frac{a_1+a_2}{2}\right)\right) - (5.4 + 0.024{a_1}^2 + 0.144a_1)\right. \\
&\qquad\qquad\qquad\qquad\left. -(5.4 + 0.024{a_2}^2 + 0.144a_2)\right) \\
&= 5.4 + 0.024a_1a_2 + 0.0144\left(\frac{a_1+a_2}{2}\right)
\end{aligned}$$

In particular, we have that $\log\left(\frac{q(51,2033)}{1-q(51,2033)}\right)$ and $\log\left(\frac{q(63,2033)}{1-q(63,2033)}\right)$ follow a multivariate normal distribution with mean -6.18, -2.46 and covariance matrix $\begin{pmatrix} 7.128 & 8.1648 \\ 8.1648 & 15.768 \end{pmatrix}$

We use a simulation to estimate the expected value of $50000q(51,2033)(0.00324 - q(63,2033))_+$. We do this by simulation

```
library(MASS)
a<-mvrnorm(1000000,c(-6.18,-2.46),rbind(c(7.128,8.1648),c(8.1648,15.768)))
q1<-exp(a[,1])/(1+exp(a[,1]))
q2<-exp(a[,2])/(1+exp(a[,2]))
50000*mean(q1*(0.00324-q2)*(q2<0.00324))
```

My simulation gives the value 0.01279491.