Generating Functions

Given a sequence a_n of numbers (which can be integers, real numbers or even complex numbers) we try to describe the sequence in as simple a form as possible. Where possible, the best way is usually to give a closed form – i.e. to express a_n as a function of n such as $a_n = 2^n - 3n + 2$ or $a_n = \binom{n}{7}$. This allows us to calculate a particular value of a_n easily, and gives us good insights that will often allow us to prove what we want to about our sequence.

Unfortunately, not all sequences can be described directly by such a formula, and in cases where they can, it is not always easy to find the formula. Therefore, in many cases we describe our sequence by a recurrence, e.g. $a_n = 3a_{n-1} + 2a_{n-2} - 7, a_0 = 2, a_1 = 1$. The problem with this is that it doesn't give us much information about the a_n – it is impractical to use this to calculate $a_{1,000,000}$ using this recurrence for example, and we do not get a clear insight into how fast the sequence grows. Also, it is difficult to see how we would change the recurrence relation to get certain related sequences – for example, given recurrences for sequences a_n and b_n, there is no obvious way to find a recurrence for the sequence $a_n + b_n$, or for sequences na_n.

Another way we could describe the sequence is to view the a_n as the coefficients of a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. [We call this a formal power series because for a general sequence a_n, we might well find that there is no non-zero value of x for which the sequence converges, so we do not get a genuine function. In practice, however, this method is most useful when we get a function $f(x)$ which we can describe explicitly.] $f(x)$ is called the *generating function* of the sequence a_n. This is a slightly abstract way to describe our sequence – saying a_n is the coefficient of x^n in the power series of $f(x)$. The justification for this method is that it works in a lot of cases. We will mostly be using generating functions as an intermediate step when we try to go from recurrences to sequences.

We will need to perform the following tasks:

1. Find the generating function for a sequence:
   - (a) given an explicit formula for the terms of the sequence by recognising the power series.
   - (b) given an explicit formula for the terms of the sequence by relating it to a power series that we know.
   - (c) given a recurrence relation for the sequence.
2. Find an explicit formula for the terms of the sequence:
   - (a) by spotting a function as one whose Taylor series we know.
   - (b) by spotting that a function is related to one whose power series we know.
   - (c) by partial fractions.

There are other things that we can do with generating functions, such as find the sum of a sequence, or the expected value of a random variable, or recurrence relations satisfied by the sequence.

1 Finding generating functions

1.1 Known Taylor series

The following sequences are examples of Taylor series of well-known functions.

- Let a_n be the constant sequence $a_n = 1$ for all $n \geqslant 0$. The generating function is $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1 < x < 1$.
- Let $a_n = \frac{1}{n!}$ for all $n \geqslant 0$. The generating function is $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.
- Let $a_n = \begin{cases} \frac{(-1)^m}{n!} & \text{if} n = 2m + 1 \text{forainteger} m \\ 0 & \text{otherwise} \end{cases}$ for all $n \geqslant 0$. The generating function is $f(x) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} = \sin x$.
- Fix m. Let $a_n = \binom{m}{n}$. The generating function is $\sum_{n=0}^{\infty} \binom{m}{n} x^n$, but for $n > m$, the binomial coefficients are 0, so the generating function is $\sum_{n=0}^{m} \binom{m}{n} x^n$, which, by the binomial theorem is $(1 + x)^m$.

1.2 Deriving new generating functions from old

There are many operations we can perform on a sequence that can be easily described in terms of its generating functions:

- We can form the sequence $a_n = \lambda^n b_n$, by substituting λx for x in the generating function of b_n, i.e. if b_n has generating function $g(x)$ then $a_n = \lambda^n b_n$ has generating function $f(x) = g(\lambda x)$.

  [More generally, we can substitute any function for x in the generating function for b_n, and this will give us another function, which will generate another sequence. However, in many cases, the sequence we get cannot be easily described from the original sequence. For example, if we substitute x^2 for x, then we get the generating function for the sequence $a_{2n} = b_n, a_{2n+1} = 0$. If the function by which we replace x is non-zero when $x = 0$, then we may not always get a convergent sum when we substitute, and thus we will not always get a sequence.]

- Given series a_n and b_n with generating functions $f(x)$ and $g(x)$ respectively, then the generating function for $\lambda a_n + \mu b_n$ for constant λ and μ is $\lambda f(x) + \mu g(x)$.
- We can shift the indices. For example, $b_n = a_{n-1}$. If $f(x)$ is the generating function for a_n then $xf(x)$ is the generating function for b_n.

- We can differentiate the power series. If $f(x)$ is the generating function for a_n then $f'(x)$ is the generating function for $b_n = (n+1)a_{n+1}$.
- We can combine these to get that if $b_n = na_n$, and $f(x)$ is the generating function for a_n, then $xf'(x)$ is the generating function for b_n.
- Given series a_n and b_n with generating functions $f(x)$ and $g(x)$ respectively, then the function $f(x)g(x)$ is the generating function for the sequence $c_k = \sum_{i=0}^k a_i b_{k-i}$.
- We can repeat this multiplication inductively – given sequences $a_{1,n}, \dots, a_{k,n}$, with generating functions $f_1(x), \dots, f_k(x)$, we can take their product $f_1(x)f_2(x)\cdots f_k(g)$. It is the generating function for the sequence $b_l = \sum_{i_1+\cdots+i_k=l} a_{1,i_1}a_{2,i_2}\cdots a_{k,i_k}$.

We can also perform the opposites of these operations, for example, integrating generating functions, but this is not as useful, and will not come up in this course.

We can use these operations with the power series that we already know to find generating functions for other sequences.

Examples 1. (i) Let $a_n = n+1$. The generating function is now given by $f(x) = \sum_{n=0}^\infty (n+1)x^n$. This power series is the derivative of $\sum_{n=0}^\infty x^{n+1} = \frac{x}{1-x}$, which is $\frac{1}{(1-x)^2}$.

(ii) Let $a_n = n$. The generating function is now given by $f(x) = \sum_{n=0}^\infty nx^n$. This is obtained from the previous example by shifting indices, which corresponded to multiplying by x, making the generaing function $\frac{x}{(1-x)^2}$.

Alternatively, we can notice that $n = (n+1)-1$, so we can get the generating function as the difference of two generating functions – i.e. $f(x) = \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{x}{(1-x)^2}$.

Another way to do this is to multiply the generating function for 1 by itself.

(iii) More generally, if we fix k, and let $a_n = \binom{n+k}{k}$, we can observe that $\binom{n+k}{k}x^n$ is the kth derivative of $\frac{x^{n+k}}{k!}$, so the generating function is $f(x) = \frac{1}{k!}\left(\frac{d}{dx}\right)^k\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^{k+1}}$.

(iv) If we fix m, and let $b_n = \binom{m}{n}$, we use the binomial theorem to get that the generating function for b_n is $g(x) = (1+x)^m$. We can differentiate this twice to get that $a_n = \binom{n+2}{2}\binom{m}{n+2}$ has generating function $f(x) = m(m-1)(1+x)^{m-2}$.

1.3 Finding generating functions from a recurrence

So far, the examples have all been sequences where we already know a simple formula for a_n, so the generating functions are not a great deal of use. We want to be able to find the generating function for a sequence given by a recurrence. We will give an example of how we can do this:

Example 1. a_n is a sequence given by $a_n = 3a_{n-1} + 7$, $a_0 = 0$. [It is not too difficult to guess the solution to this recurrence directly and prove it by induction, but for this example we will solve it by generating functions.]

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ We multiply the recurrence by x_n, and sum over all values of n for which the recurrence holds ($n \geqslant 1$) to get

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 7x^n$$

We can rewrite these sums: the sum on the LHS is just $f(x) - a_0 = f(x)$, while the first sum on the RHS is $x\sum_{m=0}^{\infty} a_m x^m = xf(x)$. The last sum on the RHS is $\sum_{n=1}^{\infty} 7x^n = \frac{7x}{1-x}$. We therefore get the equation:

$$f(x) = 3xf(x) + \frac{7x}{1-x}$$

which we can solve to get

$$f(x) = \frac{7x}{(1-x)(1-3x)}$$

The method we used for finding the generating function is as follows:

- Multiply the recurrence by x^n.
- Add the resulting equations for each n for which the recurrence is valid.
- Break up into separate sums.
- Some of the series will hopefully be related to the generating function $f(x)$ that we are looking for in a simple way. The other sums will hopefully be power series of functions we know (or can work out from functions we know by the methods in Section 1.2). We substitute what we can for all the series.
- This will give us an equation for $f(x)$. In the best case, it will be an equation such that for a fixed value of x we will get an equation in $f(x)$ that we can solve. In other cases, we may get a differential equation or a functional equation. (A differential equation is an equation involving $f'(x)$ or higher derivatives. A functional equation is one involving evaluating f at more than one point, e.g. $f(x) = xf(x^2) - 5$.) [You will not be required to solve a differential or functional equation in this course.]

Example 2. If we let T_n be the number of at most binary trees with n nodes, we can show that it satisfies the recurrence $T_n = \sum_{i=0}^{n-1} T_i T_{n-1-i}$ for $n \geqslant 1$, and $T_0 = 1$. We can multiply the recurrence by x^n, to get $T_n x^n = \sum_{i=0}^{n-1} T_i T_{n-1-i} x^n$, and add them to get $\sum_{n=1}^{\infty} T_n x^n = \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} T_i T_{n-1-i} x^n = x\sum_{n=1}^{\infty}\sum_{i=0}^{n-1} T_i T_{n-1-i} x^{n-1}$. If we let $f(x) = \sum_{n=0}^{\infty} T_n x^n$ be the generating function of T_n, then this equation becomes $f(x) - 1 = xf(x)^2$, which we can solve to get

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Example 3.

$$y_n = Ay_{n-1} + By_{n-2}$$

We start by multiplying by x^n and summing over n to get

$$\sum_{n=2}^{\infty} y_n x^n = A \sum_{n=1}^{\infty} y_n x^{n+1} + B \sum_{n=0}^{\infty} y^n x^{n+2}$$

If we let $f(x) = \sum_{n=0}^{\infty} y_n x^n$, this gives $f(x) - y_0 - y_1 x = Ax(f(x) - y_0) + Bx^2 f(x)$, or $f(x)(1 - Ax - Bx^2) = y_0(1 - Ax) + y_1$. If we let $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ be the roots of the quadratic $1 - Ax - Bx^2 = 0$, assuming they are distinct, (so α and β are the roots of $x^2 - Ax - B = 0$) this gives us

$$f(x) = \frac{y_0(1 - Ax) + y_1}{(1 - \alpha x)(1 - \beta x)}$$

2 Finding a sequence from the generating function

We have found the generating functions for some sequences given by recurrences. You may be asking yourself "What use is it?" Often, we will be able to find an explicit formula for the terms in the sequence.

Sometimes the function will be a function whose Taylor series we already know – for example e^x, $(1 + x)^\alpha$ for some α. Sometimes, it will not be exactly a Taylor series we know, but it will be closely related to one that we know.

Example 4. In the previous section, we found the generating function for T_n is $f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. We can find the power series for $\sqrt{1 - 4x}$ as a binomial: it is $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n$. To get a power series, we need the numerator to be divisible by x, so we will try $f(x) = \frac{1-\sqrt{1-4x}}{2x}$. The power series is $-\frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} (-4)^{n+1} x^n$

Observe that $\binom{\frac{1}{2}}{n}(-4)^n = \frac{1 \times -1 \times -3 \times \cdots \times 3 - 2n}{2^n n!}(-4)^n = -\frac{2^n(1 \times 1 \times 3 \times \cdots \times 2n-3)}{n!} = -2^n \frac{(2n-2)!}{2^{n-1}(n-1)!n!} = -2\frac{(2n-2)!}{(n-1)!n!}$

We therefore have that $T_n = \frac{(2(n+1)-2)!}{(n+1-1)!(n+1)!} = \frac{-1}{n+1}\binom{2n}{n}$

[The sequence T_n is a well-known sequence called the Catalan numbers. It occurs in a variety of problems in combinatorics.]

2.1 Partial Fractions

When the generating function that we find is a ratio of two polynomials, the denominator of which factors as a product of linear factors, the easiest way to find its Taylor series is the method of partial fractions – the idea is that we can express this rational function as a sum of fractions of the form $\frac{A}{1-bx}$ for some values of A and b, but these fractions are functions whose Taylor series we already know: $\frac{A}{1-bx} = \sum_{n=0}^{\infty} Ab^n x^n$, so we can get the Taylor series for their sum by just adding termwise.

We will show how we do this for the generating function $f(x) = \frac{7x}{(1-x)(1-3x)}$ that we calculated in Example 1. Finding the Taylor series by finding all the derivatives is very messy in this case. However, as we will see, the method of partial fractions is much easier. The fraction $\frac{7x}{(1-x)(1-3x)}$ can be expressed as a sum of two fractions $\frac{A}{1-x} + \frac{B}{1-3x}$ for numbers A and B to be determined. Once we have it as a sum of these fractions, we can find the Taylor series for each fraction seperately (which is easy because they are special cases of the Taylor series for the fraction $\frac{1}{1-x}$, which we already know) and then add them.

To find A and B, we simply write out the equation

$$\frac{7x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

and multiply through by $(1-x)(1-3x)$ to get rid of the fractions. This gives

$$A(1-3x) + B(1-x) = 7x$$

The easiest way to solve this is to make the substitutions $x = 1$ and $x = \frac{1}{3}$ to get $A = -\frac{7}{2}$ and $B = \frac{7}{2}$ respectively.

We can now find the Taylor series for $f(x)$ by adding these:

$$f(x) = \frac{7}{2}\sum_{n=0}^{\infty}(3x)^n - \frac{7}{2}\sum_{n=0}^{\infty} x^n = \frac{7}{2}\sum_{n=0}^{\infty}(3^n - 1)x^n$$

This gives us the explicit formula $a_n = \frac{7}{2}(3^n - 1)$.

Example 5. In Example 3, we showed that when a_n is given by the recurrence, $y_n = Ay_{n-1} + By_{n-2}$, the generating function is

$$f(x) = \frac{y_0(1-Ax) + y_1}{(1-\alpha x)(1-\beta x)}$$

where α and β are the roots of $t^2 - At - B = 0$.

We can rewrite this as a partial fraction

$$f(x) = \frac{C}{1-\alpha x} + \frac{D}{1-\beta x}$$

for some C and D.

We therefore get

$$f(x) = \sum_{n=0}^{\infty}(C\alpha^n + D\beta^n)x^n$$

which is the formula we gave in Course 2112.