Guide to writing proofs

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This handout is designed to help explain what needs to be included when writing a proof, and how proofs should be set out. The important thing is to make it completely clear to the reader exactly what you are trying to do.

While proofs will vary, depending exactly what we are trying to prove, the following points should almost always be kept in mind:

- Always be as precise and explicit as possible. No statement should be in any way ambiguous, and the reader should understand exactly what you are trying to say. It will often be clearer if you give a name to everything you mention. For example, if you are asked to prove something about a general convergent sequence, it will usually be a good idea to start with a sentence like "Let a_n be a sequence converging to a limit a."

- With the exception of other theorems that you quote, the proof should be self-contained. Always define any variables, terms and concepts that you use in the argument. Never assume that the reader has read the textbook.

- Each statement should follow from statements made earlier in the proof, definitions, assumptions made in the statement you are trying to prove, and theorems that can be quoted. It is often a good idea to make it clear exactly which of these the statement follows from.

  This means that you should work from the conditions in the statement of the theorem, to the conclusion. Sometimes when working out the details, it is easier to work backwards from the conclusion to the conditions. However, when writing out the proof you should work forwards. This may be important, as some of the steps may not be reversible.

  One exception to this: some proofs start by assuming the conclusion is false, and getting a contradiction, thereby showing that the conclusion must have been true.

- When quoting a theorem, always make sure that you know exactly what the theorem says – This may prevent you from reaching a false conclusion by using a theorem whose assumptions are not satisfied. It is best to quote the theorem by name (or number) if you know the name or number of the theorem. If not, you should make sure that the reader knows that you are quoting a theorem, and knows which one.

  One way to ensure this would be to write "Theorem:" and then write out the statement word for word. In some cases this may not be necessary, but it is better to do this when it is not necessary than to not explain adequately which theorem you are using.

- If a statement is the start of a new line of reasoning, and does not follow from the preceding statements, it is usually best to make this clear by starting it with "But", "However", "On the other hand", or some other word or phrase that indicates that the statement is not a consequence of the preceding argument.

- Sometimes, when writing a long proof, it is helpful to say early on what the general strategy will be. This will help the reader to understand the details of the proof. The proofs you may be asked to give in the exam will probably be simple enough that this will not be necessary.

- When proving statements of the form "There exists an x such that ...", it is usually best to write down x explicitly, or at least show how to find a value x which works - e.g. as the limit of some sequence.

- Similarly, if you are asked whether a statement of the form "For all x, ...", is true, and it is not true, you should usually show this by finding a value of x for which the statement does not hold. (Called a counterexample.)

- When proving a statement of the form "For all x, there is a y such that ...", you should usually give a method of calculating y (perhaps a formula) from any possible value of x.

- Some people find it helpful to explicitly write out the definitions of all the terms used in the statement of the theorem. For example, if the theorem starts with the assumption that the sequence a_n converges, you might start the proof: "a_n converges. This means that there is some a such that $(\forall \epsilon > 0)(\exists N)(\forall n \geqslant n)(|a_n - a| < \epsilon)$."

  This simultaneously defines the variable a, and tells you its properties. It will often give a hint about what to do next. For example, here, we will usually want to choose a value for $\epsilon > 0$. Then the convergence will give us a value of N, which we will use in our argument.

  It will usually be necessary to write out definitions of terms in the statement of the theorem – you will rarely be asked to prove something where you do not need to use one of the conditions. However, sometimes you will use the condition by quoting another theorem where it is a condition, and you will not need to write out the definition in this case. (Though it won't be a problem if you do.)

- Remember that looking at examples does not constitute a proof.

- Make sure you name all your variables differently, and differently from any variables that may be defined in the statement of the theorem.