Bonus Question 1

The number

$$r = \frac{\sqrt{5}-1}{2} \approx 0.618$$

is called the golden ratio. The aim of this question is to show that

$$r = \frac{\sqrt{5}-1}{2} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}.$$

a) Let

$$a_1 = 1, \quad a_2 = \frac{1}{1+a_1}, \quad a_3 = \frac{1}{1+a_2}, \quad \ldots$$

Compute the first few terms of this sequence. How is this sequence related to the *Fibonacci sequence*: 1, 1, 2, 3, 5, 8, 11, 21, ...?

b) Show that *if* the sequence $\{a_n\}$ converges, then it must necessary converge to r, i.e. $a_n \to r$ as $n \to \infty$, provided that $\lim_{n\to\infty} a_n$ exists.

c) Prove that $\lim_{n\to\infty} a_n = r$.

Note: We say that

$$\lim_{n\to\infty} a_n = r \qquad \text{or} \qquad a_n \to r \text{ as } n \to \infty$$

if and only if for every $\varepsilon > 0$ there exists an integer N such that $|a_n - r| < \varepsilon$ for all $n > N$.