Bonus question 2

1. Consider the following function:

$$f(x) = \left\{ \begin{array}{c} 0 \text{ if } x \text{ is irrational} \\ 1/q \text{ if } x \text{ is rational and } x = p/q \text{ in lowest terms} \end{array} \right. .$$

Show that

$$\lim_{x \to a} f(x) = 0 \text{ for all numbers } a \in [0,1].$$

Conclude that f is continous at all irrrational numbers and discontinous at all rational numbers in $[0,1]$.

2a. Show that at any given time, there exist two locations on the equator that are antipodal to each other and have the same temperature. Hint: you may assume that the temperature depends continuously on the location.

2b. Suppose that f is a continous function on [0,1] with $f(0) = f(1)$. Show that for any integer $n \geq 1$, there exists a number $x \in [0, 1 - \frac{1}{n}]$ such that $f\left(x + \frac{1}{n}\right) = f(x)$.

2c. Give an example of a continous function f and a number $a < \frac{1}{2}$ such that $f(0) = f(1)$ but such that $f(x + a) \neq f(x)$ for any $x \in [0, 1 - a]$.