Test 2 Practice Problems Nov '06

1 i) $f(x) = \ln\left(\frac{x}{1+x^2}\right) = \ln x - \ln(1+x^2)$

$$f'(x) = \frac{1}{x} - \frac{2x}{1+x^2} = \frac{1+x^2-2x^2}{x(1+x^2)} = \frac{1-x^2}{x(1+x^2)}$$

---

ii) $y = x^{\ln x}$

$\ln y = \ln(x^{\ln x}) = \ln x \ln x = (\ln x)^2$

Diff. w.r.t. x:

$$\frac{1}{y}\frac{dy}{dx} = 2\ln x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = 2y\frac{\ln x}{x} = \frac{2x^{\ln x}\ln x}{x}$$

---

iii) $f(x) = x\tan^{-1}x$

$$f'(x) = \frac{x}{1+x^2} + \tan^{-1}x, \quad f'(1) = \frac{1}{2} + \frac{\pi}{4}$$

---

2. $(x^2+y^2)^2 = 4x^2y$

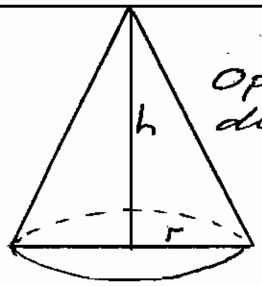
$$2(x^2+y^2)\frac{d}{dx}(x^2+y^2) = 4\left(x^2\frac{dy}{dx} + y \cdot 2x\right)$$

$$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 4\left(x^2\frac{dy}{dx} + 2xy\right)$$

At $(1,1)$, $\cancel{2}(\cancel{2})\left(2+2\frac{dy}{dx}\right) = \cancel{4}\left(\frac{dy}{dx}+2\right)$

(Slope of tangent) $\frac{dy}{dx} = 0$

3.



Optional diag.

$\frac{dV}{dt} = \frac{1}{4}$. Need $\left.\frac{dh}{dt}\right|_{h=2}$

$V = \frac{1}{3}\pi r^2 h, \quad h = 2r$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2\frac{dh}{dt}$. Subst: $\frac{1}{4} = \frac{\pi 2^2}{4}\frac{dh}{dt}$

Height incr. at $\frac{1}{4\pi}$ m/min $\quad \frac{dh}{dt} = \frac{1}{4\pi}$

4. $f(x) = x + \frac{1}{2x^2}$, $f'(x) = 1 - \frac{2}{2x^3} = 1 - \frac{1}{x^3}$

$f'(x) = 0$ when $1 = \frac{1}{x^3}$, $x^3 = 1$, $x = 1$ (Cr. pt.)

Compare $f(\frac{1}{2}) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$

$f(1) = 1 + \frac{1}{2} = \frac{3}{2}$ Abs. min

$f(3) = 3 + \frac{1}{18} = 3\frac{1}{18}$ Abs. max

---

5. $f(x) = \frac{x-2}{x+1}$, $f'(x) = \frac{(x+1).1 - (x-2).1}{(x+1)^2}$

$f'(x) = \frac{3}{(x+1)^2}$ Never zero. No cr. pts.

Compare $f(0) = -2$ Abs. min

$f(9) = \frac{7}{10}$ Abs. max

6i) $\lim_{x \to -\infty} x^2 e^x = \lim_{x \to -\infty} \frac{x^2}{e^{-x}} \overset{H}{=} \lim_{x \to -\infty} \frac{2x}{-e^{-x}}$

[Type '$\infty . 0$'   Type '$\frac{\infty}{\infty}$'   Type '$\frac{\infty}{\infty}$']

$\overset{H}{=} \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$

---

ii) $\lim_{x \to 0} \frac{5^x - 3^x}{x} \overset{H}{=} \lim_{x \to 0} \frac{5^x \ln 5 - 3^x \ln 3}{1}$

[Type '$\frac{0}{0}$'] $= \ln 5 - \ln 3 = \ln \frac{5}{3}$

---

iii) $\lim_{x \to \infty} x^{1/x}$: Put $y = x^{1/x}$

$\ln y = \ln (x^{1/x}) = \frac{1}{x} \ln x$

But $\lim_{x \to \infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$

$\ln y \to 0$ so $y \to 1$ i.e. $\lim_{x \to \infty} x^{1/x} = 1$

7\. i) $\frac{d}{dx}\ln(3x) = \frac{3}{3x} = \frac{1}{x}$

ii) $\frac{d}{dx}\ln(e^x) = \frac{e^x}{e^x} = 1$

or use $\ln(e^x) = x$

iii) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

8\. $f(x) = x^4 + 3x^3 - x + 2$, $f'(x) = 4x^3 + 9x^2 - 1$

$f''(x) = 12x^2 + 18x = 6x(2x+3)$

$f''(x) = 0$ at $x = 0, -\frac{3}{2}$

Interval	$6x$	$2x+3$	$f''(x)$	$f(x)$
$(-\infty, -\frac{3}{2})$	$-$	$-$	$+$	CU
$(-\frac{3}{2}, 0)$	$-$	$+$	$-$	CD
$(0, \infty)$	$+$	$+$	$+$	CU

Infl. pts. at $x = 0, -\frac{3}{2}$

9\. $f(x) = x^2e^x$, $f'(x) = x^2e^x + 2xe^x$
$= e^x x(x+2)$

$f'(x) = 0$ when $x = 0, -2$

Interval	e^x	x	$x+2$	$f'(x)$	$f(x)$
$(-\infty, -2)$	$+$	$-$	$-$	$+$	Incr
$(-2, 0)$	$+$	$-$	$+$	$-$	Decr
$(0, \infty)$	$+$	$+$	$+$	$+$	Incr.

Max at $(-2, \frac{4}{e^2})$, min. at $(0,0)$

10\. $f'(x) = \frac{(1+x)^2 \cdot 1 - x \cdot 2(1+x)}{(1+x)^4} = \frac{(1+x)[(1+x) - 2x]}{(1+x)^4}$

$= \frac{1-x}{(1+x)^3}$

$f'(x) = 0$ at $x = 1$. (Cr. #)

$x = -1$ is not in domain of $f(x)$.

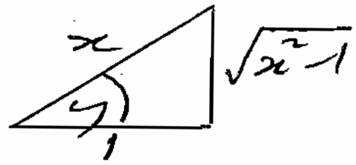
Interval	$(1-x)$	$(1+x)^3$	$f'(x)$	$f(x)$
$(-\infty, -1)$	$+$	$-$	$-$	Decr.
$(-1, 1)$	$+$	$+$	$+$	Incr.
$(1, \infty)$	$-$	$+$	$-$	Decr.

Max. at $(1, \frac{1}{4})$

11. $y = \sec^{-1} x$.

$\sec y = x$. Diff. w.r.t. x:

$\sec y \tan y \dfrac{dy}{dx} = 1$



$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$ for y in 1^{st}, 3^{rd} quadrants where both $\sec y$ and $\tan y$ are positive.

[See p 235, # 54]

---

12. $f(x) = \sqrt[3]{x}$ $\qquad f(27) = 3$

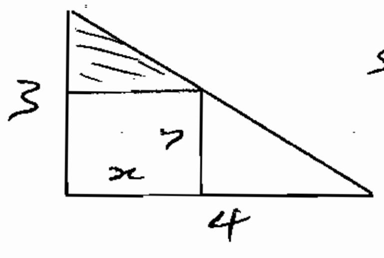
$f'(x) = \dfrac{1}{3x^{2/3}}$ $\qquad f'(27) = \dfrac{1}{3 \cdot 9} = \dfrac{1}{27}$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$L(25) = 3 - \frac{2}{27} = \frac{79}{27}$$

$$f(25) = \sqrt[3]{25} \approx L(25) = \frac{79}{27}.$$

13.



Similar Δs: $\dfrac{x}{3-y} = \dfrac{4}{3}$

$$3x = 4(3-y)$$

$$y = \frac{12-3x}{4}$$

Area $A = xy$

$$= \frac{x}{4}(12 - 3x) = \frac{3x}{4}(4-x)$$

Maximise $A = f(x) = \dfrac{3x}{4}(4-x)$

on $[0, 4]$.